Tutorial:

Magnon-magnon interactions in spin-wave theory:

Spontaneous decay, renormalization & implications for topological excitations

Jeffrey G. Rau

University of Windsor

SPICE-Workshop on Quantum Geometry and Transport of Collective Excitations in (Non-)Magnetic Insulators – May 8th, 2025

University of Windsor

NSERC CRSNG

Tutorial:

Magnon-magnon interactions in spin-wave theory:

Spontaneous decay, renormalization & implications ~~for topological excitations~~

Jeffrey G. Rau

University of Windsor

University of Windsor

NSERC CRSNG

SPICE-Workshop on Quantum Geometry and Transport of Collective Excitations in (Non-)Magnetic Insulators – May 8th, 2025

Outline

1. History
2. Linear spin-waves
3. Non-linear spin waves
4. Applications

Outline

1. ~~History~~

2. Linear spin-waves

3. Non-linear spin waves

4. Applications

Some Useful References

Colloquium: Spontaneous magnon decays, M. E. Zhitomirsky and A. L. Chernyshev, Rev. Mod. Phys. **85**, 219 (2013)

Spin waves in a triangular lattice antiferromagnet: Decays, spectrum renormalization, and singularities, A. L. Chernyshev and M. E. Zhitomirsky, Phys. Rev. B **79**, 144416 (2009)

Dynamical structure factor of the triangular-lattice antiferromagnet, M. Mourigal, W. T. Fuhrman, A. L. Chernyshev, and M. E. Zhitomirsky, Phys. Rev. B **88**, 094407 (2013)

Quantum Many-Particle Physics, Negele and Orland (1998)

Spin Waves, Akhiezer, A. I., V. G. Bar'yakhtar, and S. V. Peletminskii (1968)

Spin Waves, J. Van Kranendonk and J. H. Van Vleck, Rev. Mod. Phys. **30**, 1 (1958)

…

Linear Spin Waves

Review: *Spin-wave Theory*

- Consider a general (bilinear) spin model in *classical* limit ($1/S = 0$)

$$H = \frac{1}{2}\sum_{rr'} \boldsymbol{S}_r \boldsymbol{J}_{rr'} \boldsymbol{S}_{r'}$$

- *Find a classical ground state*: unit vectors $\boldsymbol{z}_r$ such that $\boldsymbol{S}_r = S\,\boldsymbol{z}_r$ minimizes the energy

Goal: Describe the physics *near* the $1/S = 0$ where quantum and/or thermal fluctuations allow the spins to weakly deviate from the classical ordering direction

Local Basis

- Can define *local frame* using this ordering: a right-handed (mutually orthogonal) coordinate system at each site

$$\hat{\boldsymbol{x}}_r \quad \hat{\boldsymbol{y}}_r \quad \hat{\boldsymbol{z}}_r$$

- Work in frame aligned with local frame at each site

$$\boldsymbol{S}_r = S_r^+ \hat{\boldsymbol{e}}_{r,-} + S_r^- \hat{\boldsymbol{e}}_{r,+} + S_r^z \hat{\boldsymbol{e}}_{r,0}$$

$$\hat{\boldsymbol{e}}_{r,\pm} = (\hat{\boldsymbol{x}}_r \pm i\hat{\boldsymbol{y}}_r)/\sqrt{2}$$
$$\hat{\boldsymbol{e}}_{r,0} = \hat{\boldsymbol{z}}_r$$

- Spin Hamiltonian is re-expressed as

$$= \frac{1}{2} \sum_{rr'} \left[\mathcal{J}_{rr'}^{00} S_r^0 S_{r'}^0 + \left(\mathcal{J}_{rr'}^{++} S_r^- S_{r'}^- + \mathcal{J}_{rr'}^{+0} S_r^- S_{r'}^0 + \mathcal{J}_{rr'}^{0+} S_r^0 S_{r'}^- + \mathcal{J}_{rr'}^{+-} S_r^- S_{r'}^+ + \text{h.c.} \right) \right]$$

Longitudinal terms *Transverse terms* *Transverse-Longitudinal terms*

Holstein-Primakoff Expansion

- Bosonic representation of spin operators with respect to classical ground state

Transverse part

$$\boldsymbol{S}_r = \sqrt{S}\left[\left(1 - \frac{n_r}{2S}\right)^{1/2} a_r \hat{\boldsymbol{e}}_{r,-} + a_r^\dagger \left(1 - \frac{n_r}{2S}\right)^{1/2} \hat{\boldsymbol{e}}_{r,+}\right] + (S - n_r)\hat{\boldsymbol{e}}_{r,0}$$

Longitudinal part

- where we have that $[a_r, a_{r'}^\dagger] = \delta_{rr'}$ and $n_r \equiv a_r^\dagger a_r$
- **Non-linear** in boson number
- Natural expansion parameter - *boson occupancy*

$$\frac{n_r}{2S} \ll 1$$

Linear Spin-Wave Theory

- Leading order expansion – drop the roots

$$\left(1 - \frac{n_r}{2S}\right)^{1/2} \approx 1$$

- Representation

$$\boldsymbol{S}_r \approx \sqrt{S}\left(a_r \hat{\boldsymbol{e}}_{r,-} + a_r^\dagger \hat{\boldsymbol{e}}_{r,+}\right) + (S - n_r)\hat{\boldsymbol{e}}_{r,0}$$

- Bosons represent spin raising/lowering operators in direction *transverse to the ordering*

Linear Spin-Wave Theory

- Plug this back into Hamiltonian and keep only leading terms in $1/S$

$$H = S^2 E_0 + S H_2 + \dots$$

- Constant part at $O(S^2)$ is classical energy $\quad E_0 = \frac{1}{2}\sum_{rr'} \hat{z}_r^{\top} \boldsymbol{J}_{rr'} \hat{z}_{r'}$
- Quadratic boson Hamiltonian at O(S)

$$S H_2 = \sum_{rr'} \left[\underbrace{A_{rr'}}_{\substack{\text{“Hopping”} \\ \text{like terms}}} a_r^\dagger a_{r'} + \frac{1}{2!}\left(\underbrace{B_{rr'}}_{\substack{\text{“Pairing”} \\ \text{like terms}}} a_r^\dagger a_{r'}^\dagger + \text{h.c.} \right) \right]$$

Linear Spin-Wave Theory

- In Fourier space this becomes

$$S\,H_2 = \sum_{\boldsymbol{k}} \sum_{\alpha\alpha'} \left[A_{\boldsymbol{k}}^{\alpha\alpha'} a_{\boldsymbol{k}\alpha}^{\dagger} a_{\boldsymbol{k}\alpha'} + \frac{1}{2!} \left(B_{\boldsymbol{k}}^{\alpha\alpha'} a_{\boldsymbol{k}\alpha}^{\dagger} a_{-\boldsymbol{k}\alpha'}^{\dagger} + \text{h.c.} \right) \right]$$

- The matrices given by

$$A_{\boldsymbol{k}}^{\alpha\alpha'} = S \left(\mathcal{J}_{\boldsymbol{k},\alpha\alpha'}^{+-} - \delta_{\alpha\alpha'} \sum_{\mu} \mathcal{J}_{\boldsymbol{0},\alpha\mu}^{00} \right)$$

$$B_{\boldsymbol{k}}^{\alpha\alpha'} = S\,\mathcal{J}_{\boldsymbol{k}}^{++}$$

Group into enlarged matrix

$$\boldsymbol{M}_{\boldsymbol{k}} = \begin{pmatrix} \boldsymbol{A}_{\boldsymbol{k}} & \boldsymbol{B}_{\boldsymbol{k}} \\ \boldsymbol{B}_{-\boldsymbol{k}}^{*} & \boldsymbol{A}_{-\boldsymbol{k}}^{*} \end{pmatrix}$$

Linear Spin-Wave Theory

- Stability of classical ground state restricts spin-wave Hamiltonian

Condition #1: Linear terms in bosons must vanish

- Since the ground state is an extremum of the energy

Condition #2: Matrix $\boldsymbol{M_k}$ must be positive definite

- Since the ground state is a *minimum* of the energy

- Linear terms can be removed by shift of bosons
- Interaction effects can sometimes allow expansion about *unstable* ground state

Linear Spin-Wave Theory

- Can diagonalize by Bogoliubov transformation

Diagonalize

$$\boldsymbol{\sigma}_3 \boldsymbol{M}_{\boldsymbol{k}} = \begin{pmatrix} \boldsymbol{A}_{\boldsymbol{k}} & \boldsymbol{B}_{\boldsymbol{k}} \\ -\boldsymbol{B}^*_{-\boldsymbol{k}} & -\boldsymbol{A}^*_{-\boldsymbol{k}} \end{pmatrix} \Rightarrow \boldsymbol{\sigma}_z \boldsymbol{M}_{\boldsymbol{k}} \boldsymbol{V}_{\boldsymbol{k}\alpha} = \epsilon_{\boldsymbol{k}\alpha} \boldsymbol{V}_{\boldsymbol{k}\alpha}$$

Eigenvalue

Eigenvector

- Yields independent new bosonic degrees of freedom

$$\sum_{\boldsymbol{k}\alpha} \epsilon_{\boldsymbol{k}\alpha} \left(\gamma^\dagger_{\boldsymbol{k}\alpha} \gamma_{\boldsymbol{k}\alpha} + \frac{1}{2} \right)$$

- Ground state energy at *O(S)* acquires zero-point correct; zero γ bosons in the ground state

Beyond Linear Spin-Wave Theory

- Many reasons why we often need to go beyond LSWT

Quantitative comparisons: Need more accurate spin-wave dispersion for comparison to other model calculations or experimental data

Intrinsic interactions effects: Physics of interest only appears at next to leading order *(pseudo-Goldstone gaps, spontaneous decay)*

Transport properties: Many require interactions to avoid unphysical results due to non-interacting particles *(thermal conductivity)*

Non-linear Spin Waves

Magnon-Magnon Interactions

- Continue to next order in *O(1/S)*

$$\left(1 - \frac{n_r}{2S}\right)^{1/2} \approx 1 - \frac{n_r}{4S} + \ldots$$

- Will introduce cubic and quartic interactions between the magnons
- **General expressions are quite complicated**

Will show you the formalism for simpler case:

- One sublattice with inversion symmetry at zero temperature

$$a_{\boldsymbol{k}\alpha} \to a_{\boldsymbol{k}} \qquad \boldsymbol{k} \leftrightarrow -\boldsymbol{k} \qquad T = 0$$

- Leading order parts are O(S)$^{1/2}$ and O(S)

$$H_2 = \sum_{r_1 r_2}\left[A_{r_1 r_2} a^\dagger_{r_1} a_{r_2} + \frac{1}{2!}\left(B_{r_1 r_2} a^\dagger_{r_1} a^\dagger_{r_2} + \bar{B}_{r_1 r_2} a_{r_1} a_{r_2}\right)\right],$$

$$H_3 = \frac{1}{2!}\sum_{r_1 r_2 r_3}\left[T_{r_1 r_2 r_3} a^\dagger_{r_1} a^\dagger_{r_2} a_{r_3} + \bar{T}_{r_1 r_2 r_3} a^\dagger_{r_3} a_{r_2} a_{r_1}\right],$$

$$H_4 = \sum_{r_1 r_2 r_3 r_4}\left[\frac{1}{(2!)^2} V_{r_1 r_2 r_3 r_4} a^\dagger_{r_1} a^\dagger_{r_2} a_{r_3} a_{r_4} + \frac{1}{3!}\left(D_{r_1 r_2 r_3 r_4} a^\dagger_{r_1} a^\dagger_{r_2} a^\dagger_{r_3} a_{r_4} + \bar{D}_{r_1 r_2 r_3 r_4} a^\dagger_{r_4} a_{r_3} a_{r_2} a_{r_1}\right)\right]$$

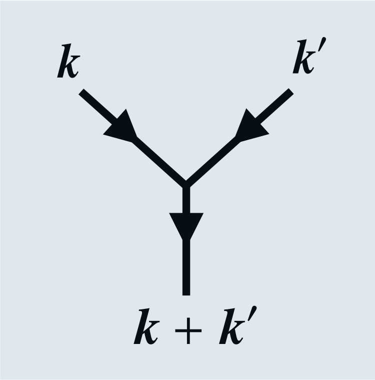
Symmetrization factors simplify perturbative expansion

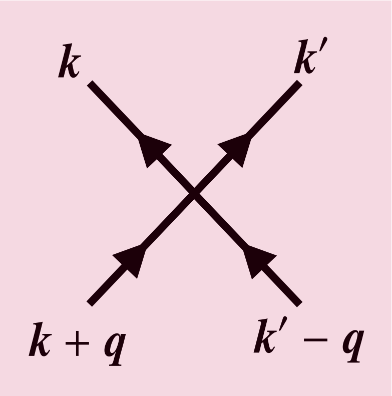
- Interactions **do not conserve boson number** (from D and T)
- Interactions **do not conserve boson number mod 2** (from T)

- More natural in Fourier space; momentum conserved in each interaction

$$H_2 = \sum_{\boldsymbol{k}} \left[A_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{1}{2!} \left(B_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a^\dagger_{-\boldsymbol{k}} + \text{h.c.} \right) \right]$$

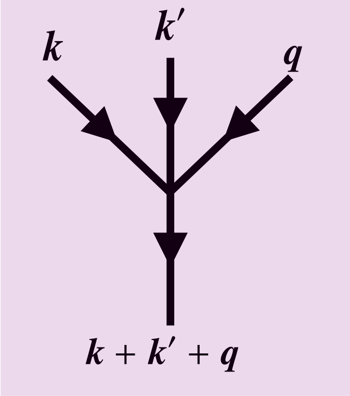
$$H_3 = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}\boldsymbol{k}'} \left(T_{\boldsymbol{k}\boldsymbol{k}'} a^\dagger_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}'} a_{\boldsymbol{k}+\boldsymbol{k}'} + \text{h.c.} \right)$$

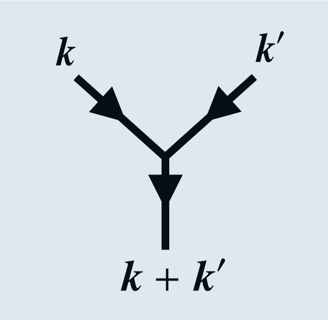


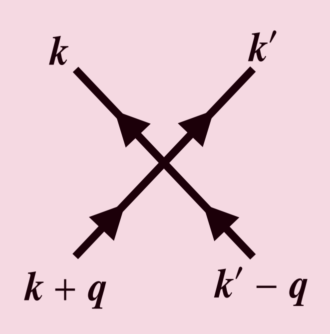


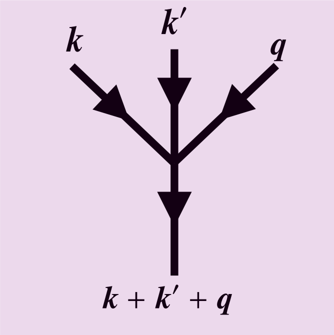
$$H_4 = \frac{1}{N} \sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} \frac{1}{(2!)^2} V_{\boldsymbol{k}\boldsymbol{k}'[\boldsymbol{q}]} a^\dagger_{\boldsymbol{k}+\boldsymbol{q}} a^\dagger_{\boldsymbol{k}'-\boldsymbol{q}} a_{\boldsymbol{k}} a_{\boldsymbol{k}'}$$

$$H_4' = \frac{1}{N} \sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} \frac{1}{3!} \left(D_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} a^\dagger_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}'} a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{q}} + \text{h.c.} \right)$$









$$A_{\boldsymbol{k}} = S\left(\mathcal{J}^{+-}_{\boldsymbol{k}} - \mathcal{J}^{00}_{\boldsymbol{0}}\right)$$

$$B_{\boldsymbol{k}} = S\,\mathcal{J}^{++}_{\boldsymbol{k}}$$

$$T_{\boldsymbol{k}\boldsymbol{k}'} = -S^{1/2}\left(\mathcal{J}^{+0}_{\boldsymbol{k}} + \mathcal{J}^{+0}_{\boldsymbol{k}'}\right)$$

$$V_{\boldsymbol{k}\boldsymbol{k}'[\boldsymbol{q}]} = \frac{1}{2}\left(\mathcal{J}^{00}_{\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q}} + \mathcal{J}^{00}_{\boldsymbol{k}'-\boldsymbol{k}-\boldsymbol{q}} + \mathcal{J}^{00}_{\boldsymbol{q}} + \mathcal{J}^{00}_{-\boldsymbol{q}}\right) - \frac{1}{2}\left(\mathcal{J}^{+-}_{\boldsymbol{k}} + \mathcal{J}^{+-}_{\boldsymbol{k}'} + \mathcal{J}^{+-}_{\boldsymbol{k}'-\boldsymbol{q}} + \mathcal{J}^{+-}_{\boldsymbol{k}+\boldsymbol{q}}\right)$$

$$D_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} = -\frac{1}{2}\left(\mathcal{J}^{++}_{\boldsymbol{k}} + \mathcal{J}^{++}_{\boldsymbol{k}'} + \mathcal{J}^{++}_{\boldsymbol{q}}\right)$$

- Vertices are (somewhat) simple functions of local exchange matrices

Free theory

- Use (diagrammatic) many-body perturbation theory about free limit

$$H_0 = \sum_{\boldsymbol{k}} \left[A_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{1}{2!} \left(B_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a^\dagger_{-\boldsymbol{k}} + \text{h.c.} \right) \right]$$

- Need **Green's function** for this free theory
- Work in *imaginary time*

$$a_{\boldsymbol{k}}(\tau) \equiv e^{\tau H_0} a_{\boldsymbol{k}} e^{-\tau H_0}$$

- Consider **normal** and **anomalous** propagators

Free Theory

(Imaginary) time ordered

$$g^{-+}(\boldsymbol{k},\omega) \equiv \int_0^\beta d\tau e^{i\omega\tau} \langle \mathcal{T} a_{\boldsymbol{k}}(\tau) a^\dagger_{\boldsymbol{k}}(0) \rangle_0$$

$$g^{+-}(\boldsymbol{k},\omega) \equiv \int_0^\beta d\tau e^{i\omega\tau} \langle \mathcal{T} a^\dagger_{\boldsymbol{k}}(\tau) a_{\boldsymbol{k}}(0) \rangle_0$$

Normal parts

$$g^{++}(\boldsymbol{k},\omega) \equiv \int_0^\beta d\tau e^{i\omega\tau} \langle \mathcal{T} a^\dagger_{\boldsymbol{k}}(\tau) a^\dagger_{\boldsymbol{k}}(0) \rangle_0$$

$$g^{--}(\boldsymbol{k},\omega) \equiv \int_0^\beta d\tau e^{i\omega\tau} \langle \mathcal{T} a_{\boldsymbol{k}}(\tau) a_{\boldsymbol{k}}(0) \rangle_0$$

Anomalous parts

Organize into 2 by 2 matrix mirroring the $\boldsymbol{M}_k$

$$\boldsymbol{g}(\boldsymbol{k},\omega) \equiv \begin{pmatrix} g^{-+}(\boldsymbol{k},\omega) & g^{--}(\boldsymbol{k},\omega) \\ g^{++}(\boldsymbol{k},\omega) & g^{+-}(\boldsymbol{k},\omega) \end{pmatrix}$$

Free Theory

$$\begin{pmatrix} a_{\boldsymbol{k}}(\tau) \\ a^{\dagger}_{-\boldsymbol{k}}(\tau) \end{pmatrix} = e^{-\tau\sigma_z M_{\boldsymbol{k}}} \begin{pmatrix} a_{\boldsymbol{k}} \\ a^{\dagger}_{-\boldsymbol{k}} \end{pmatrix}$$

- Can calculate explicitly (assume inversion so that $A_{\boldsymbol{k}} = A_{-\boldsymbol{k}}$)

$$\boldsymbol{g}(\boldsymbol{k}, \omega) = \left[-i\omega\boldsymbol{\sigma}_z + \boldsymbol{M}_{\boldsymbol{k}}\right]^{-1}$$

- Alternatively:

$$g^{-+}(\boldsymbol{k}, \omega) = g^{+-}(-\boldsymbol{k}, -\omega) = \frac{|u_{\boldsymbol{k}}|^2}{-i\omega + \epsilon_{\boldsymbol{k}}} + \frac{|v_{\boldsymbol{k}}|^2}{-i\omega + \epsilon_{\boldsymbol{k}}}$$

$$g^{--}(\boldsymbol{k}, \omega) = g^{++}(\boldsymbol{k}, \omega)^* = \frac{2\epsilon_{\boldsymbol{k}} u_{\boldsymbol{k}} v^*_{\boldsymbol{k}}}{\omega^2 + \epsilon^2_{\boldsymbol{k}}}$$

$$\epsilon_{\boldsymbol{k}} = \sqrt{A^2_{\boldsymbol{k}} - |B_{\boldsymbol{k}}|^2}$$

Green's Function

$$G^{-+}(\boldsymbol{k},\omega) \equiv \int_0^\beta d\tau e^{i\omega\tau} \langle \mathcal{T} a_{\boldsymbol{k}}(\tau) a_{\boldsymbol{k}}^\dagger(0) \rangle$$

$$G^{+-}(\boldsymbol{k},\omega) \equiv \int_0^\beta d\tau e^{i\omega\tau} \langle \mathcal{T} a_{\boldsymbol{k}}^\dagger(\tau) a_{\boldsymbol{k}}(0) \rangle$$

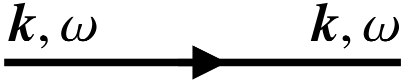
$$G^{++}(\boldsymbol{k},\omega) \equiv \int_0^\beta d\tau e^{i\omega\tau} \langle \mathcal{T} a_{\boldsymbol{k}}^\dagger(\tau) a_{\boldsymbol{k}}^\dagger(0) \rangle$$

$$G^{--}(\boldsymbol{k},\omega) \equiv \int_0^\beta d\tau e^{i\omega\tau} \langle \mathcal{T} a_{\boldsymbol{k}}(\tau) a_{\boldsymbol{k}}(0) \rangle$$

- Compute these Green's functions perturbatively in the interactions
- Only *two* of these are independent
  - $G^{+-}(\boldsymbol{k},\omega)$ and $G^{-+}(\boldsymbol{k},\omega)$ are related by symmetry
  - $G^{++}(\boldsymbol{k},\omega)$ and $G^{--}(\boldsymbol{k},\omega)$ are related by symmetry

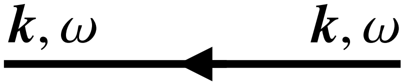
Diagrammatic Rules

- Compute perturbative contributions diagrammatically
  - Green's function can be computed as sum of diagrams
- Since have normal and anomalous Green's function have *four kinds of free propagators*



$\equiv g^{-+}(\boldsymbol{k}, \omega)$

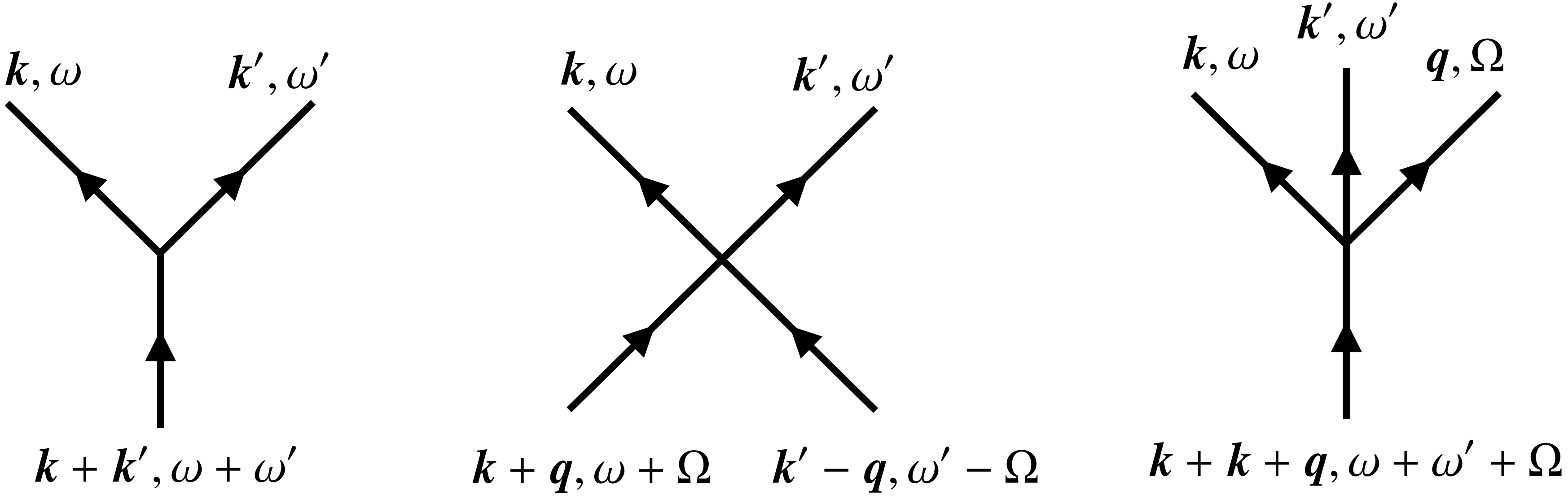
$\equiv g^{++}(\boldsymbol{k}, \omega)$



$\equiv g^{+-}(\boldsymbol{k}, \omega)$

$\equiv g^{--}(\boldsymbol{k}, \omega)$

- Three kinds of interaction vertices: two particle scattering, two magnon decay and three magnon decay



$$-\frac{1}{\sqrt{\beta N}}T^*_{\boldsymbol{k}\boldsymbol{k}'} \qquad -\frac{1}{\beta N}V_{\boldsymbol{k}\boldsymbol{k}'[\boldsymbol{q}]} \qquad -\frac{1}{\beta N}D^*_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}}$$

1. Draw all distinct connected diagrams consisting of n_4 four point vertices (of type V, D), n_3 three point vertices (of type T), connected by normal or anomalous propagators. Order is $n_4 + n_3/2$

2. For each line assign a momentum and Matsubara frequency, consistent with conservation of wave-vector and frequency at each vertex and sums over them

3. For each line include the correct free Green's function from $\boldsymbol{g}(\boldsymbol{k},\omega)$

4. For each vertex include the appropriate factor of V, D, T . Vertices with more outgoing lines than incoming lines always have a conjugation.

5. Compute the symmetry factor σ and multiply by 1/σ

Overcounting of permutations

6. For each bundle of m lines starting at the same vertex with a m_1 -fold redundancy and ending at the same vertex with a m_2-fold redundancy multiply by $1/[m!(m_1 - m)!(m_2 - m)!]$.

Often easier just to manually count contractions

Finally, let $a_\alpha|\phi\rangle = 0$ for all annihilation operators. Show that $\hat{A}_i\hat{A}_j$ is the propagator $\langle\phi|T\hat{A}_i\hat{A}_j|\phi\rangle$ and that

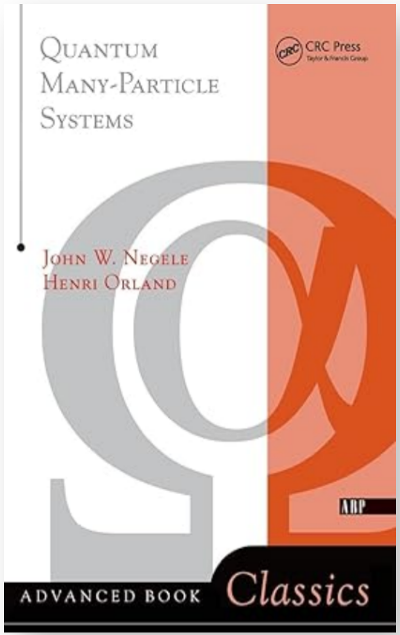
$$\langle\phi|T\hat{A}_1\hat{A}_2\cdots\hat{A}_n|\phi\rangle = \sum_{\substack{\text{all complete}\\ \text{contractions}}} \hat{A}_1\hat{A}_2\ldots\hat{A}_n$$

where

$$\hat{A}_1\hat{A}_2 = \langle\phi|T\hat{A}_1\hat{A}_2|\phi\rangle \quad .$$

PROBLEM 2.9 Generalize the Hugenholtz diagram rules to the case in which V contains one-body, two-body, ... up to n-body interactions. The principal issue is what to do about m-tuples of equivalent lines. Note that in general an m-tuple of equivalent lines could start at an n_1-body vertex and terminate at an n_2-body vertex and that multiple m-tuples could originate or terminate at the same vertex.

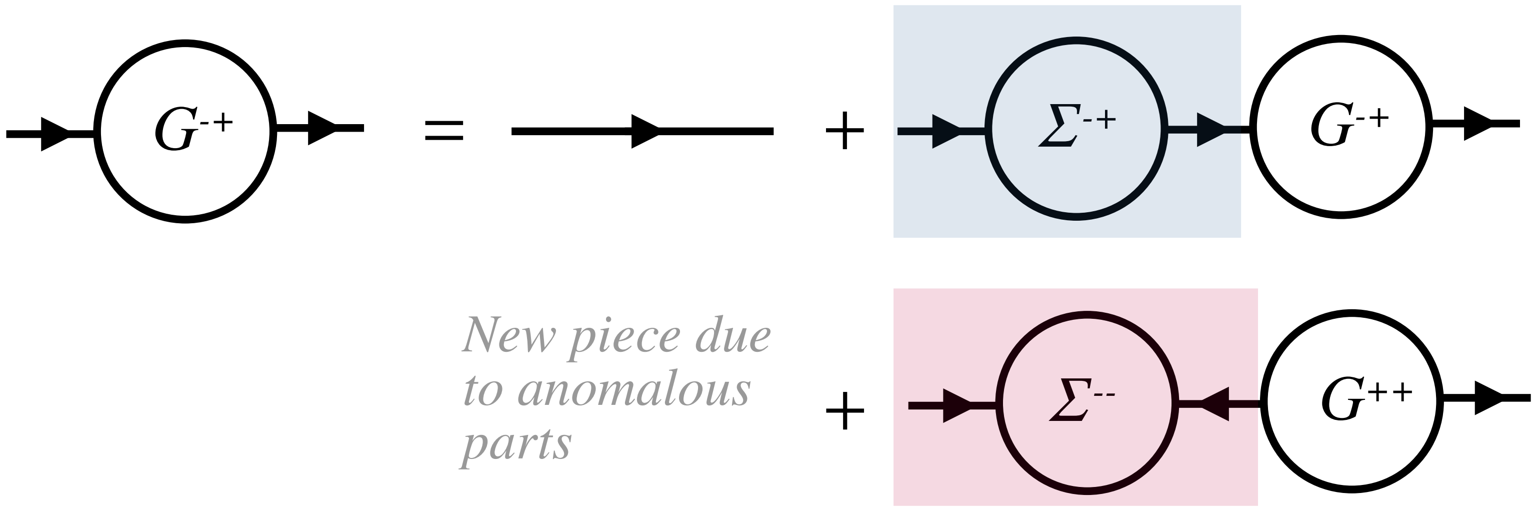
PROBLEM 2.10 **The Linked-Cluster Theorem.** Show that the exponential of the sum of all linked graphs yields the sum of all diagrams with the appropriate factors using unlabeled Feynman diagrams. First, let a_c denote the contribution of some linked cluster and consider a graph composed of n of these clusters. Explain why the symmetry factor is $n!s_c^n$ where s_c is the symmetry factor in a_c. Hence, show that the contribution of any number (including 0) of clusters a_c is $\sum_{n=0}^{\infty}\frac{a_c^n}{n!}e^{a_c}$. Generalize to show that $e^{\sum_c a_c}$ yields the sum of all diagrams with the correct symmetry factors.



Negele & Orland, *Quantum Many-Particle Systems*

Self-Energy

- Resummation of diagrams ala Dyson/Gorkov gives **self-energy**



- Self-energy and Dyson series *also* has normal and anomalous parts

Self-Energy

- Doing this for all Green's functions gives the form

$$\boldsymbol{G}(\boldsymbol{k},\omega) \equiv \left[-i\omega\boldsymbol{\sigma}_z + \boldsymbol{M}_{\boldsymbol{k}} + \boldsymbol{\Sigma}(\boldsymbol{k},\omega)\right]^{-1}$$

- Self-energy is now a 2 by 2 matrix like Green's function

$$\boldsymbol{\Sigma}(\boldsymbol{k},\omega) \equiv \begin{pmatrix} \Sigma^{-+}(\boldsymbol{k},\omega) & \Sigma^{--}(\boldsymbol{k},\omega) \\ \Sigma^{++}(\boldsymbol{k},\omega) & \Sigma^{+-}(\boldsymbol{k},\omega) \end{pmatrix}$$

- **The self-energy will be the focus of our calculations**

Continuing to Real Time

- To extract information about the magnons, we need to analytically continue to **real-time**
- Effectively substitution of frequency $i\omega \to \omega + i0^+$
- Retarded Green's function given by

$$\boldsymbol{G}_R(\boldsymbol{k}, \omega) \equiv -\boldsymbol{G}(\boldsymbol{k}, -i\omega + 0^+) = \left[\omega \boldsymbol{\sigma}_z - \boldsymbol{M}_{\boldsymbol{k}} - \boldsymbol{\Sigma}_R(\boldsymbol{k}, \omega)\right]^{-1}$$

- Similar definition for the retarded self-energy

$$\boldsymbol{\Sigma}_R(\boldsymbol{k}, \omega) \equiv -\boldsymbol{\Sigma}(\boldsymbol{k}, -i\omega + 0^+)$$

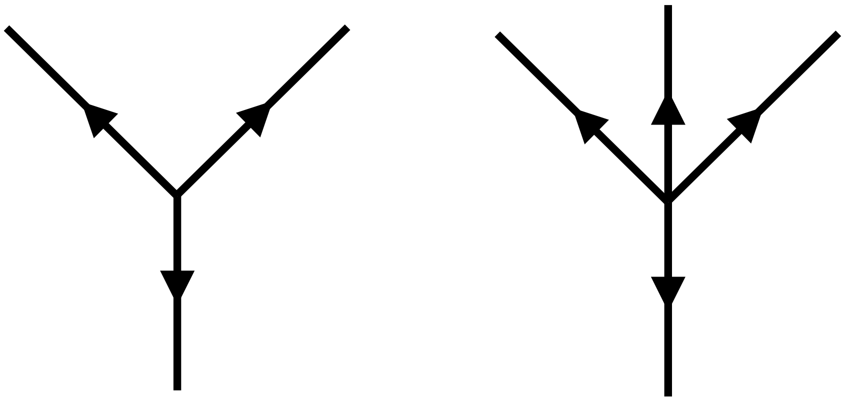
Aside: *Choice of Basis*

- I showed you this in *original* basis of magnons
- This perturbation theory (usually) is carried out in the *diagonalized basis*

Practically:

- Write in terms of $\gamma_{\boldsymbol{k}}$
- Normal order the operators
- Perturbation theory but with larger set of vertices

New vertices



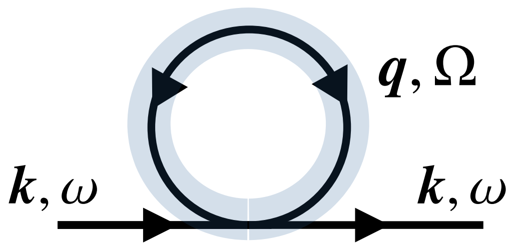
$\gamma_{\boldsymbol{k}}\gamma_{\boldsymbol{k}'}\gamma_{-\boldsymbol{k}-\boldsymbol{k}'}$ $\qquad$ $\gamma_{\boldsymbol{k}}\gamma_{\boldsymbol{k}'}\gamma_{\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{q}}\gamma_{-\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{q}}$

$\gamma_{\boldsymbol{k}}\gamma_{-\boldsymbol{k}}$

E.g. from normal ordering

Zhitomirsky & Chernyshev, RMP (2013)

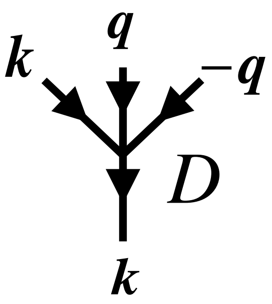
Example: *Normal Self-Energy*



$$g^{++}(\boldsymbol{k}, \omega) = \frac{2u_{\boldsymbol{k}}^* v_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}}}{\omega^2 + \epsilon_{\boldsymbol{k}}^2}$$

$$\equiv -\frac{1}{2} \cdot \frac{1}{\beta N} \sum_{\boldsymbol{q}\Omega} g^{++}(\boldsymbol{q}, \Omega) D_{\boldsymbol{k},\boldsymbol{q},-\boldsymbol{q}}$$

Perform sum at T=0



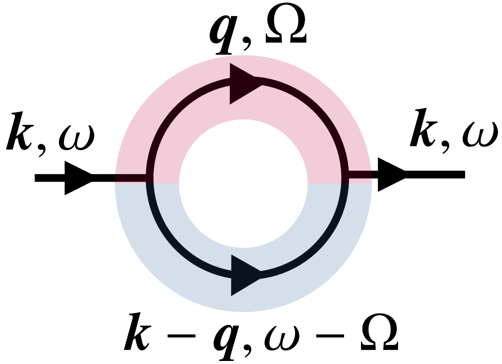
$$= -\frac{1}{2} \sum_{\boldsymbol{q}} \langle a_{\boldsymbol{q}}^\dagger a_{-\boldsymbol{q}}^\dagger \rangle_0 D_{\boldsymbol{k},\boldsymbol{q},-\boldsymbol{q}}$$

Analytic continue to real time

$$\rightarrow +\frac{1}{2} \sum_{\boldsymbol{q}} \langle a_{\boldsymbol{q}}^\dagger a_{-\boldsymbol{q}}^\dagger \rangle_0 D_{\boldsymbol{k},\boldsymbol{q},-\boldsymbol{q}}$$

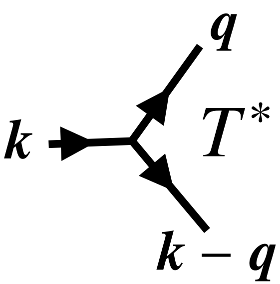
Example: *Normal Self-Energy*

$$g^{-+}(\boldsymbol{k},\omega) = \frac{|u_{\boldsymbol{k}}|^2}{-i\omega + \epsilon_{\boldsymbol{k}}} + \frac{|v_{\boldsymbol{k}}|^2}{i\omega + \epsilon_{\boldsymbol{k}}}$$



$$\equiv \frac{1}{2} \cdot \frac{1}{\beta N} \sum_{\boldsymbol{q}\Omega} g^{-+}(\boldsymbol{q},\Omega) g^{-+}(\boldsymbol{k}-\boldsymbol{q},\omega-\Omega) T^*_{\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q}} T_{\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q}}$$

Evaluate sum at T=0



$$= \frac{1}{2N} \sum_{\boldsymbol{q}} |T_{\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q}}|^2 \left[\frac{|u_{\boldsymbol{q}}|^2 |u_{\boldsymbol{k}-\boldsymbol{q}}|^2}{-i\omega + \epsilon_{\boldsymbol{q}} + \epsilon_{\boldsymbol{k}-\boldsymbol{q}}} + \frac{|v_{\boldsymbol{q}}|^2 |v_{\boldsymbol{k}-\boldsymbol{q}}|^2}{i\omega + \epsilon_{\boldsymbol{q}} + \epsilon_{\boldsymbol{k}-\boldsymbol{q}}} \right]$$

To real time

$$= \frac{1}{2N} \sum_{\boldsymbol{q}} |T_{\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q}}|^2 \left[\frac{|u_{\boldsymbol{q}}|^2 |u_{\boldsymbol{k}-\boldsymbol{q}}|^2}{\omega - \epsilon_{\boldsymbol{q}} - \epsilon_{\boldsymbol{k}-\boldsymbol{q}} + i0^+} - \frac{|v_{\boldsymbol{q}}|^2 |v_{\boldsymbol{k}-\boldsymbol{q}}|^2}{\omega + \epsilon_{\boldsymbol{q}} + \epsilon_{\boldsymbol{k}-\boldsymbol{q}} + i0^+} \right]$$

Quasi-particle poles

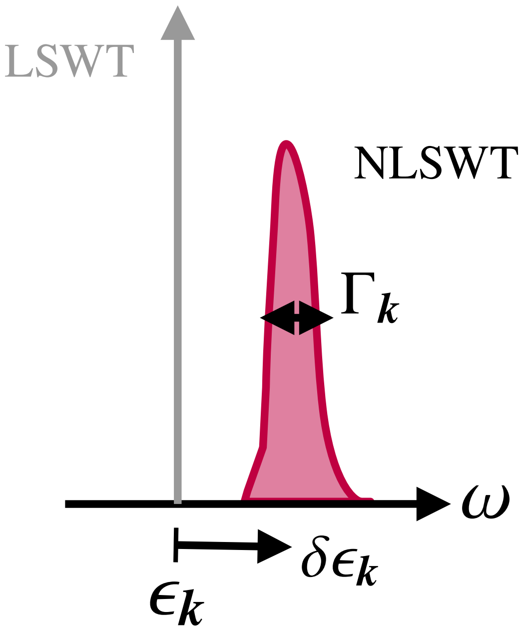
- Excitations appear as **poles**

$$\det\left[\omega \boldsymbol{\sigma}_z - \boldsymbol{M}_{\boldsymbol{k}} - \boldsymbol{\Sigma}_R(\boldsymbol{k}, \omega)\right] = 0$$

- If interactions are well-behaved can solve via perturbation theory

$$\omega = \epsilon_{\boldsymbol{k}} + \boldsymbol{V}_{\boldsymbol{k}}^{\dagger} \boldsymbol{\Sigma}_R(\boldsymbol{k}, \epsilon_{\boldsymbol{k}}) \boldsymbol{V}_{\boldsymbol{k}}$$

- Solution for poles can be **complex**; real and imaginary part

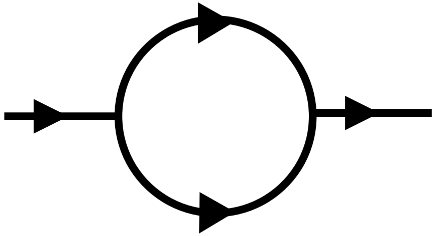


$$\delta\epsilon_{\boldsymbol{k}} = \mathrm{Re}[\boldsymbol{V}_{\boldsymbol{k}}^{\dagger} \boldsymbol{\Sigma}_R(\boldsymbol{k}, \epsilon_{\boldsymbol{k}}) \boldsymbol{V}_{\boldsymbol{k}}]$$

$$\Gamma_{\boldsymbol{k}} = -\mathrm{Im}[\boldsymbol{V}_{\boldsymbol{k}}^{\dagger} \boldsymbol{\Sigma}_R(\boldsymbol{k}, \epsilon_{\boldsymbol{k}}) \boldsymbol{V}_{\boldsymbol{k}}]$$

Spontaneous Decay

- *When* do we get an imaginary contribution?
- Non-zero if frequency in two-magnon continuum



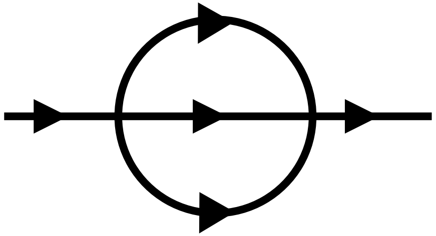
$$\mathrm{Re}\Sigma^{-+}(\boldsymbol{k},\omega) = \frac{1}{2N}\mathcal{P}\sum_{\boldsymbol{q}} \frac{|T_{\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q}}|^2}{\omega - \epsilon_{\boldsymbol{q}} - \epsilon_{\boldsymbol{k}-\boldsymbol{q}}}$$

Non-zero when $\omega = \varepsilon_q + \varepsilon_{q-k}$

Set $u_k = 1$, $v_k = 0$ for simplicity

$$-\mathrm{Im}\Sigma^{-+}(\boldsymbol{k},\omega) = \frac{1}{2N}\sum_{\boldsymbol{q}} |T_{\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q}}|^2 \boxed{\delta(\omega - \epsilon_{\boldsymbol{q}} - \epsilon_{\boldsymbol{k}-\boldsymbol{q}})}$$

- If *T* vertex vanishes, leading process can be $O(D^2)$



Dynamical Structure Factor

- Computing the dynamical structure factor not *entirely* from the magnon Green's function

Just G^{-+}

Renormalize/mix the usual Green's functions

$$\langle S_r^+(\tau) S_{r'}^-(0)\rangle_c \approx S\,\langle a_r(\tau) a_{r'}^\dagger(0)\rangle - \frac{1}{4}\Big(\langle a_r(\tau) a_{r'}^\dagger(0) n_{r'}(0)\rangle + \langle n_r(\tau) a_r(\tau) a_{r'}^\dagger(0)\rangle\Big)$$

$$\langle S_r^z(\tau) S_{r'}^z(0)\rangle_c \approx \langle n_r(\tau) n_{r'}(0)\rangle$$

Density-density: Probes two-magnon properties

$$\langle S_r^z(\tau) S_{r'}^+(0)\rangle_c \approx -S^{1/2}\langle n_r(\tau) a_{r'}^\dagger(0)\rangle$$

Renormalize/mix the usual Green's functions ***&*** *involve "bubbles"*

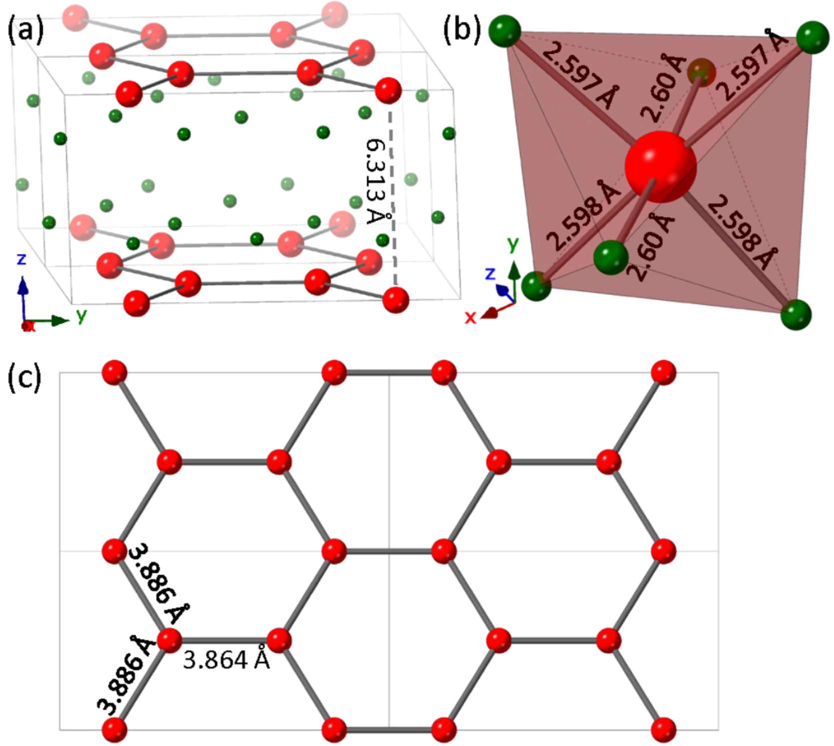
Mourigal *et al*, PRB (2013)

Effects of Interactions

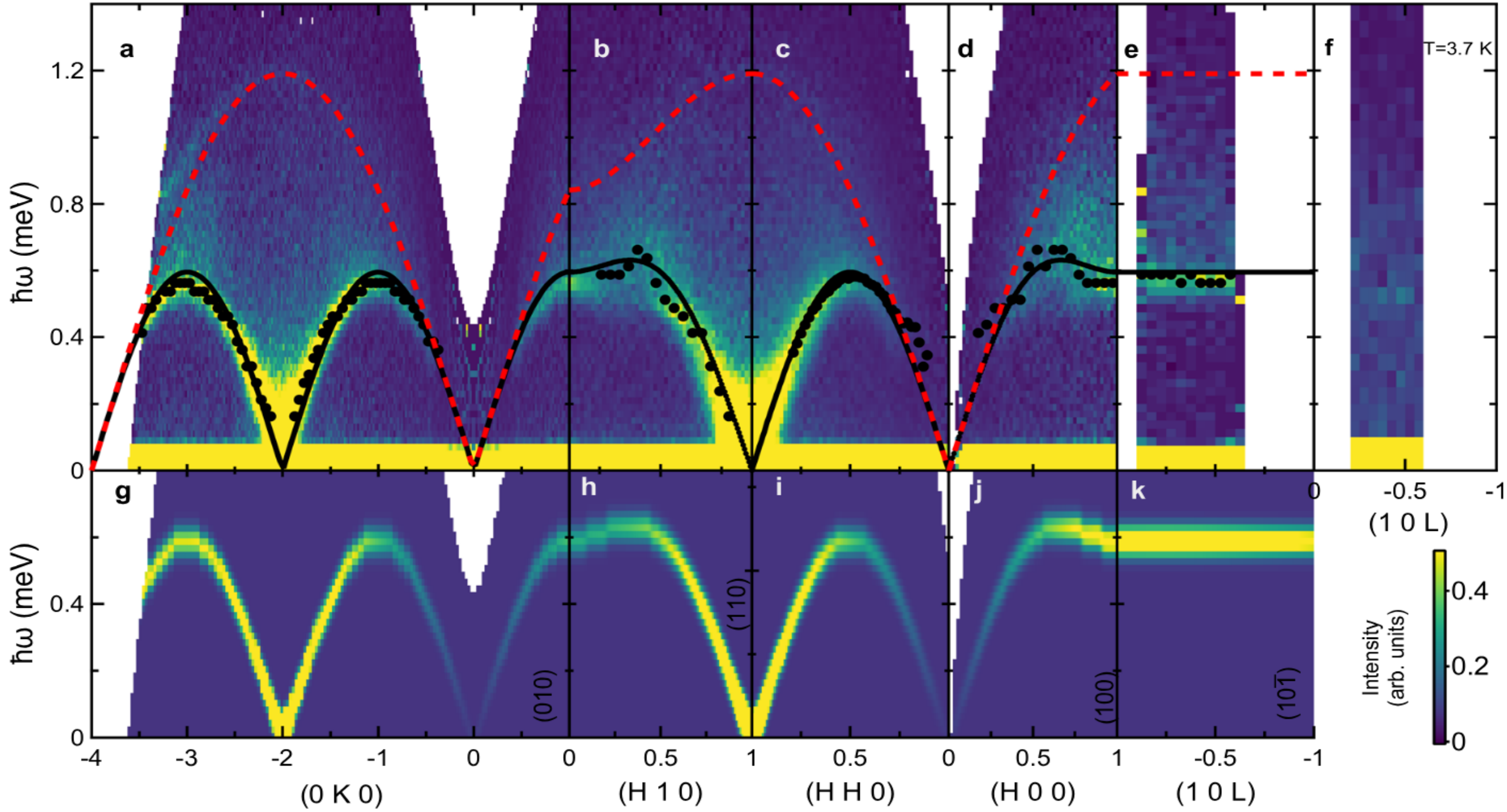
Example: *Renormalization*

- Rare-earth magnet: $YbCl_3$
- Honeycomb lattice antiferromagnet ($S_{\mathrm{eff}} = 1/2$)
- Neel order below ~0.6 K
- **Surprisingly isotropic exchange interactions**
- Model with Heisenberg exchange

$$H = \frac{1}{2} \sum_{rr'} J_{rr'} \boldsymbol{S}_r \cdot \boldsymbol{S}_{r'}$$



Xing et al, Phys. Rev. B **102**, 014427 (2020)
Sala *et al*, Phys. Rev. B **100**, 180406 (2019);
Rau & Gingras, Phys. Rev. B **98**, 054408 (2018)



Fit using linear spin-wave theory

$J = 0.421$ meV

Sala *et al*, Nat. Comm. **12**, 171 (2021)

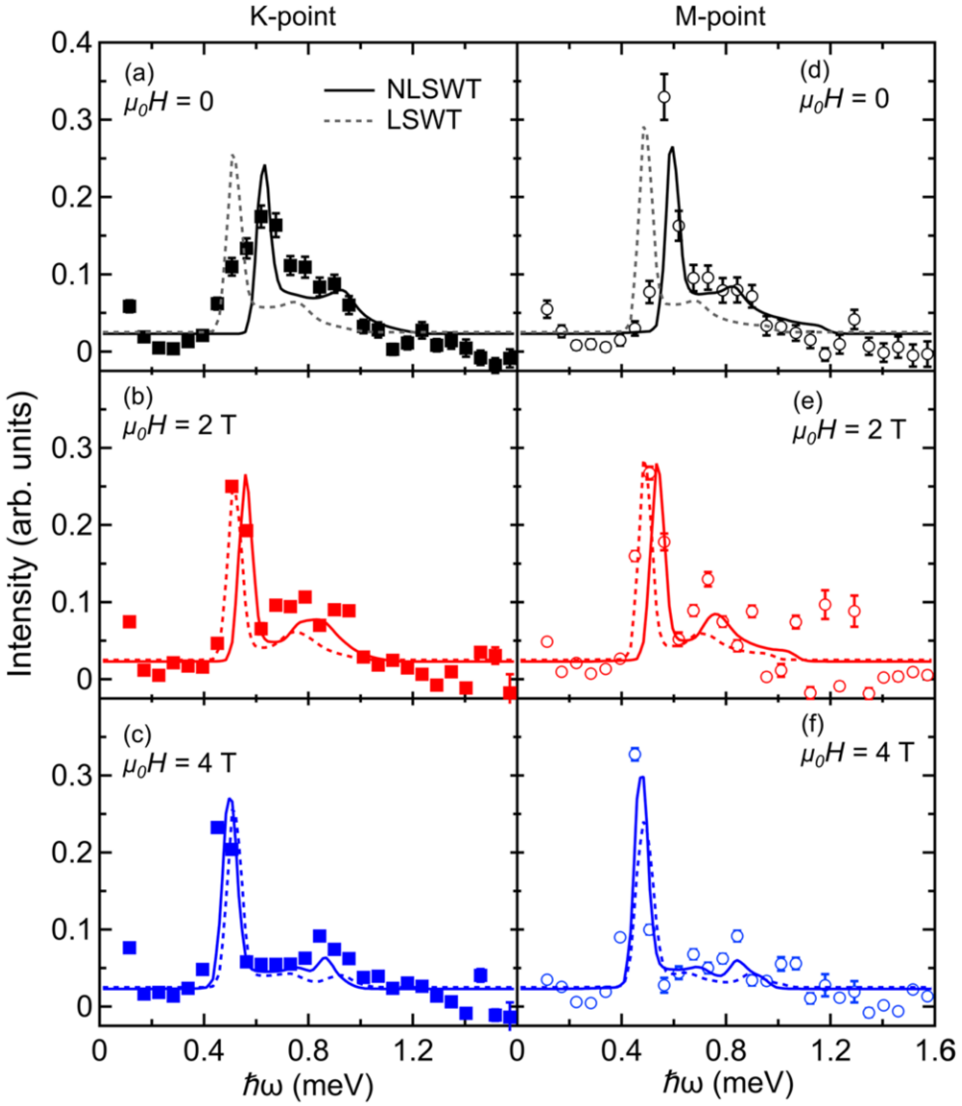
- Non-linear effects renormalize spectrum
- At leading order rescales

$$\epsilon_{\boldsymbol{k}} + \delta\epsilon_{\boldsymbol{k}} \approx Z\epsilon_{\boldsymbol{k}}$$

- Factor is approximately $Z \sim 1.3$
- If fit with NLSWT would find

$$J = 0.344 \text{ meV}$$

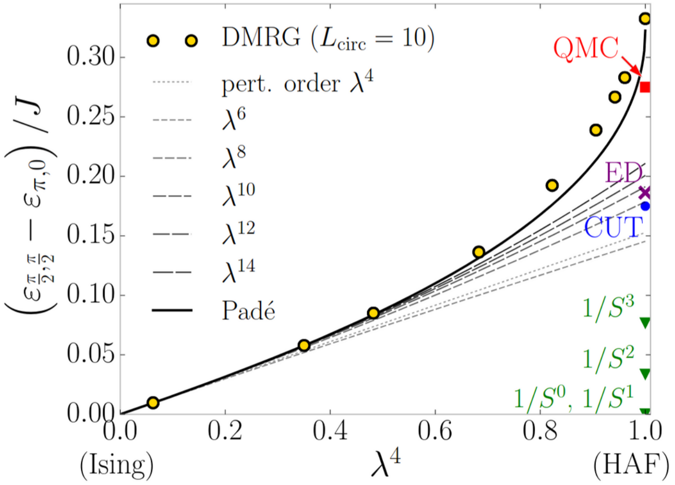
- Validated by high-field experiment; LSWT becomes exact when $|\boldsymbol{B}|/J$ large

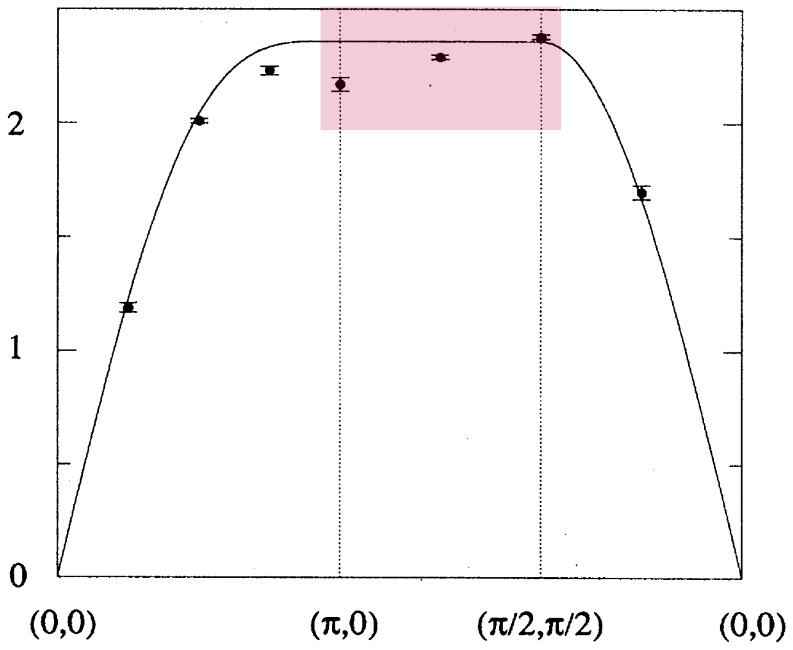


Sala *et al*, Comm. Phys. **6**, 234 (2023)

Example: *Renormalization*

- … some well known cases where interactions aren't enough
- Square lattice Heisenberg antiferromagnet



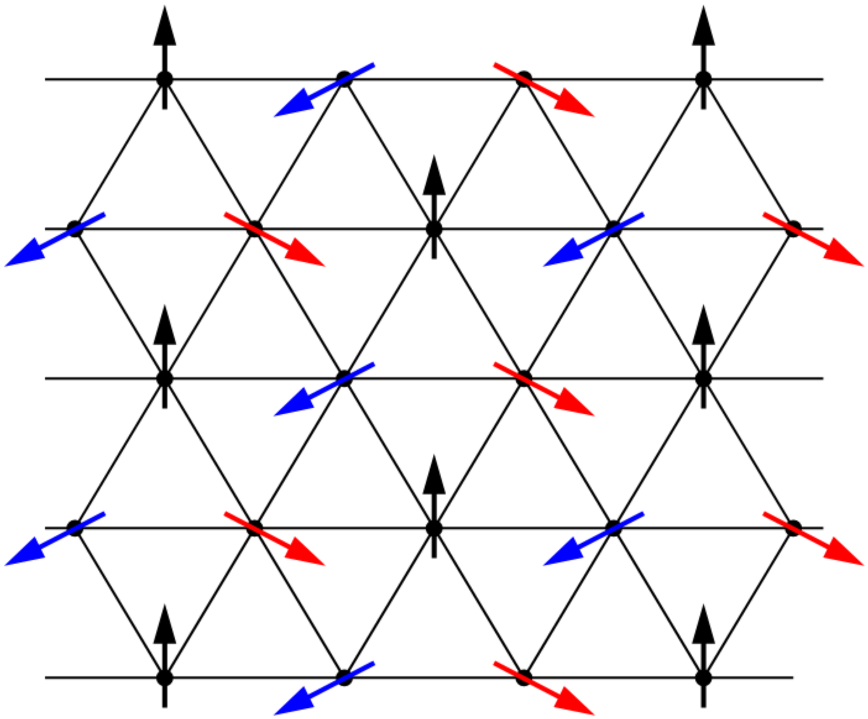


- Spin-waves are **flat** along line at leading order in interactions

Singh & Gelfand, Phys. Rev. B **52**, R15695 (1995); Verresen *et al*. Phys Rev. B **98**, 155102 (2018)

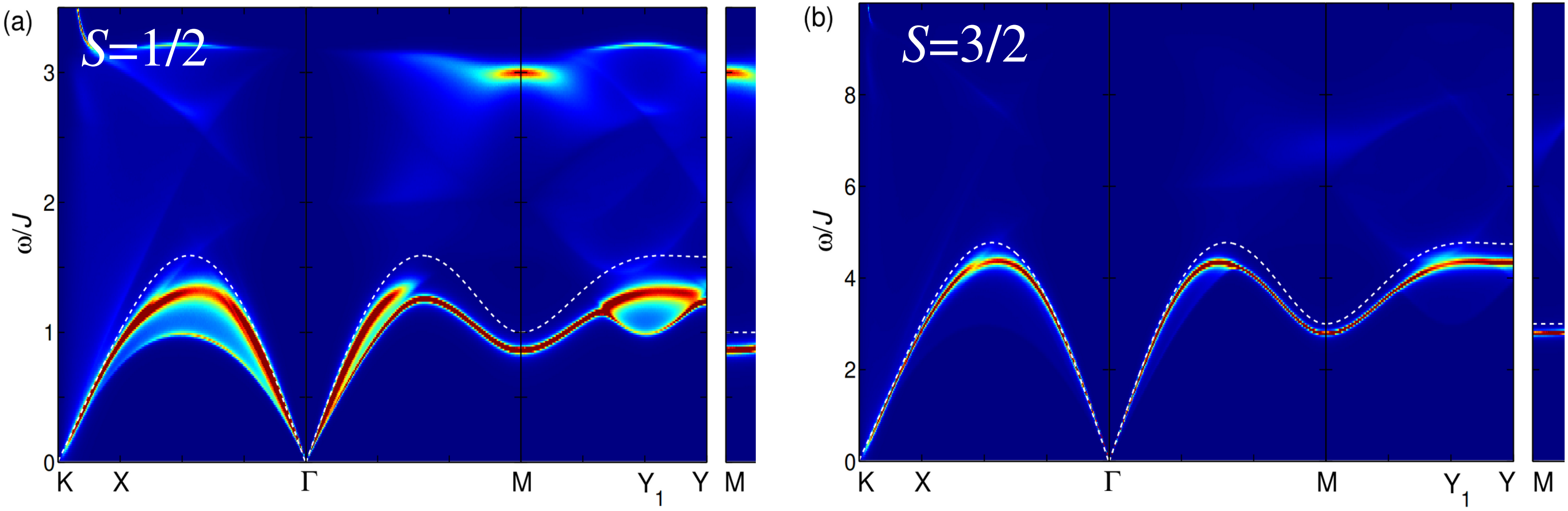
Example: *Spontaneous Decay*

- Triangular Heisenberg anti-ferromagnet
- Isotropic interactions, but non-collinear order
- **Spontaneous magnon decay is allowed**
- Presence of gapless Goldstone mode means kinematic condition not too hard to satisfy



Zhitomirsky & Chernyshev, PRB (2009)

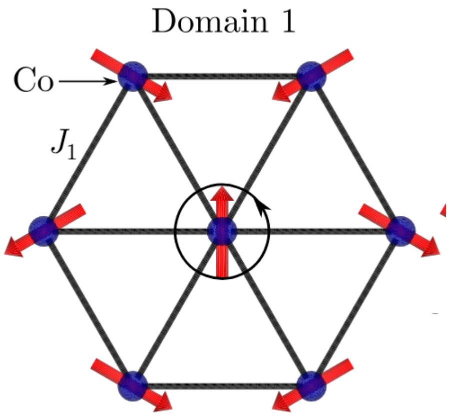
Example: *Spontaneous Decay*

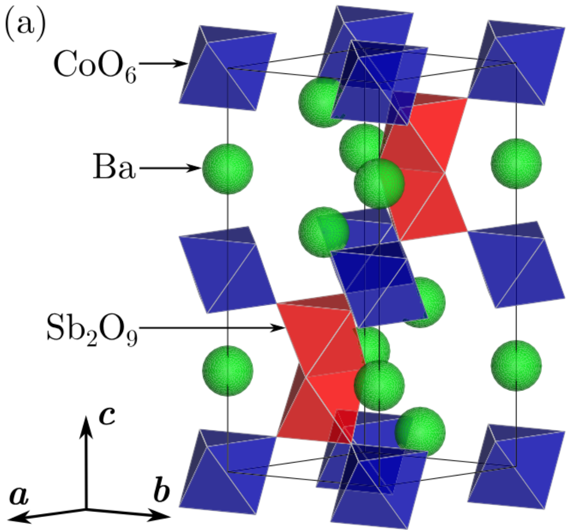


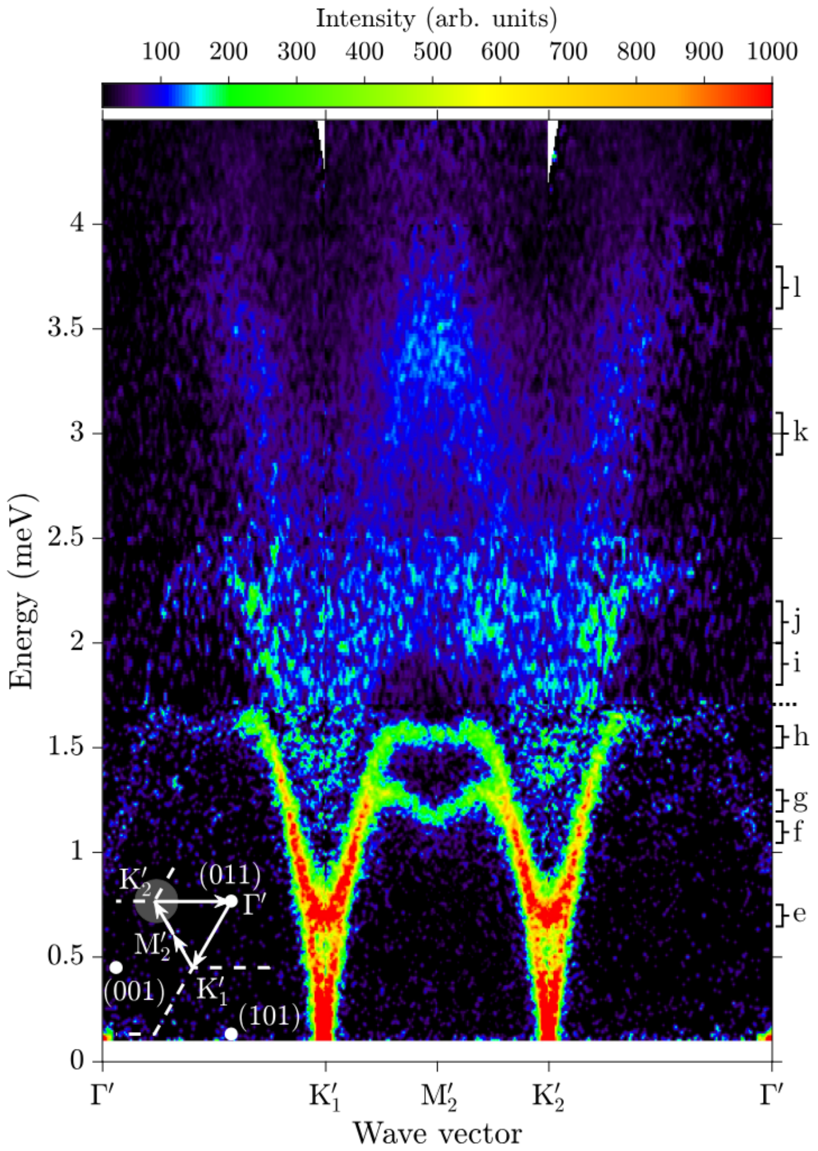
- Significant decay at small *S*; approaches LSWT as *S* increases

Mourigal *et al*, PRB (2013)

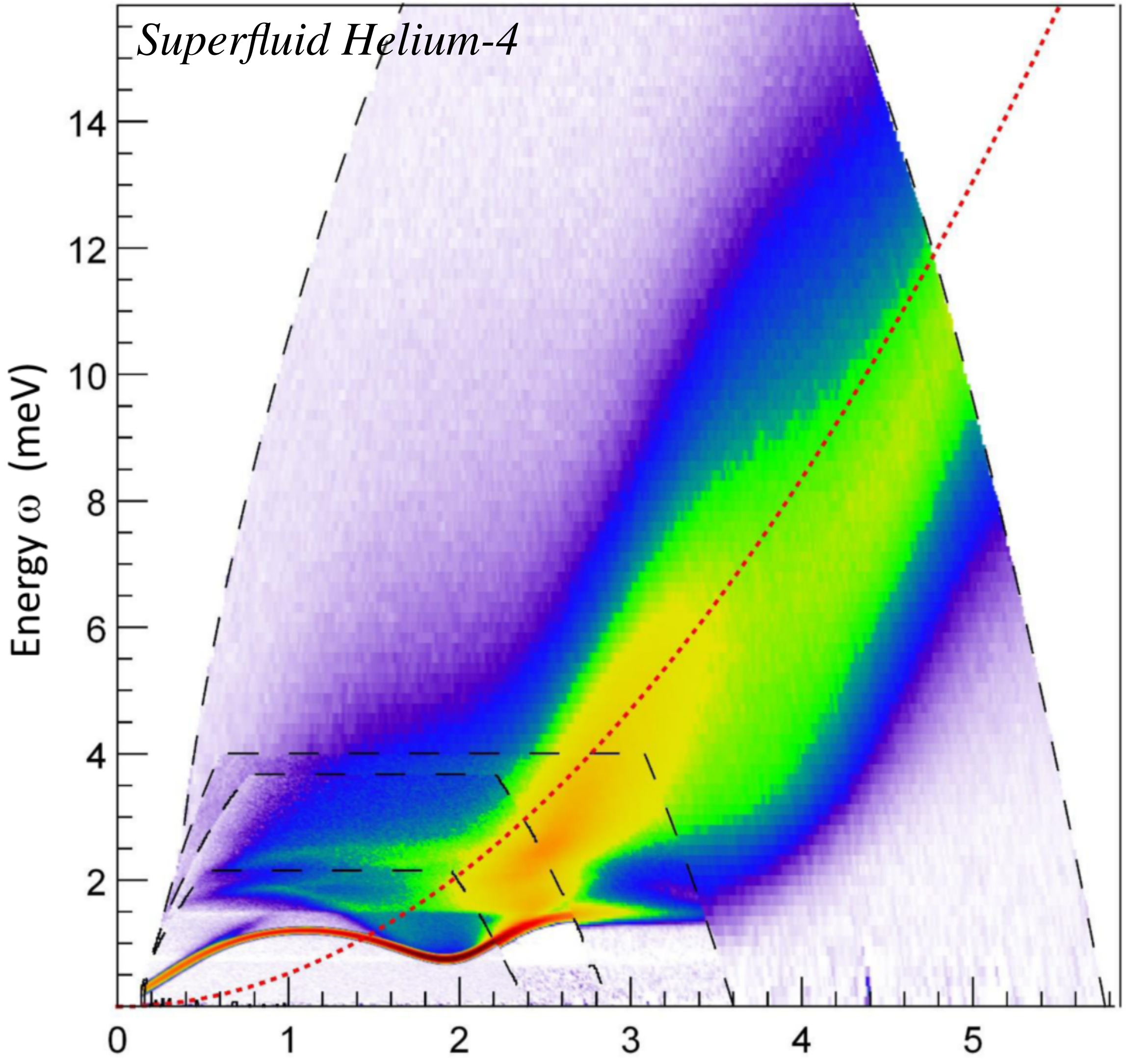
- Not always apparent in real materials
- $Ba_3CoSb_2O_9$ is (near) ideal triangular S=1/2 HAFM
- **Quasi-particle decay (mostly) avoided**

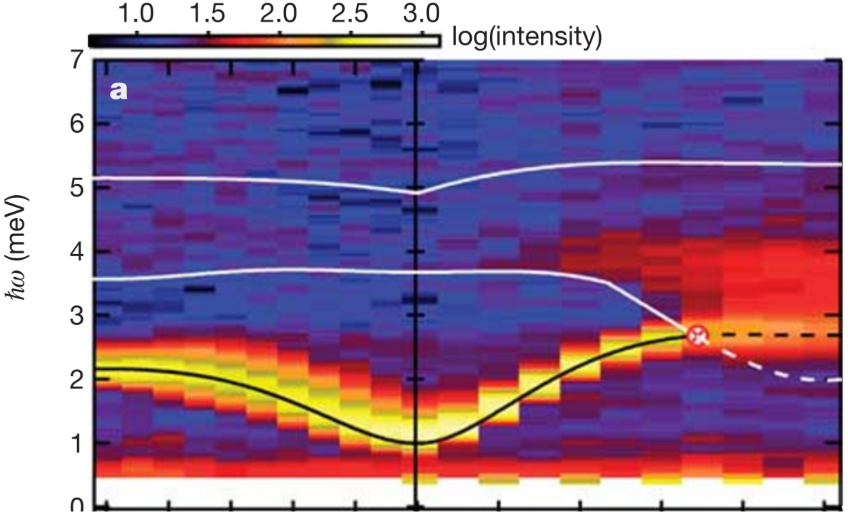






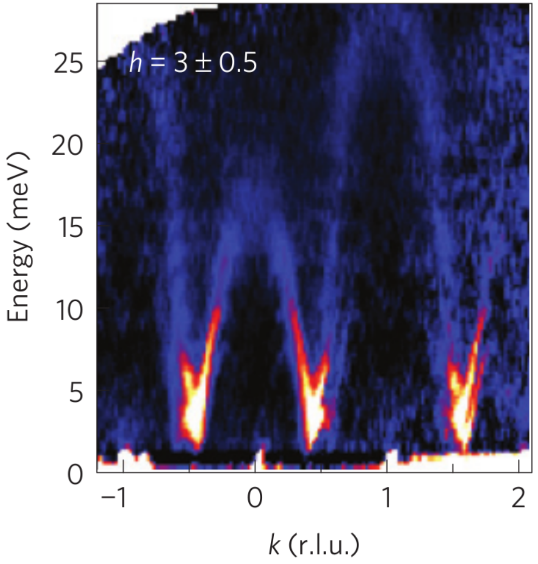
Macdougal *et al*, Phys. Rev. B **102**, 064421 (2020)



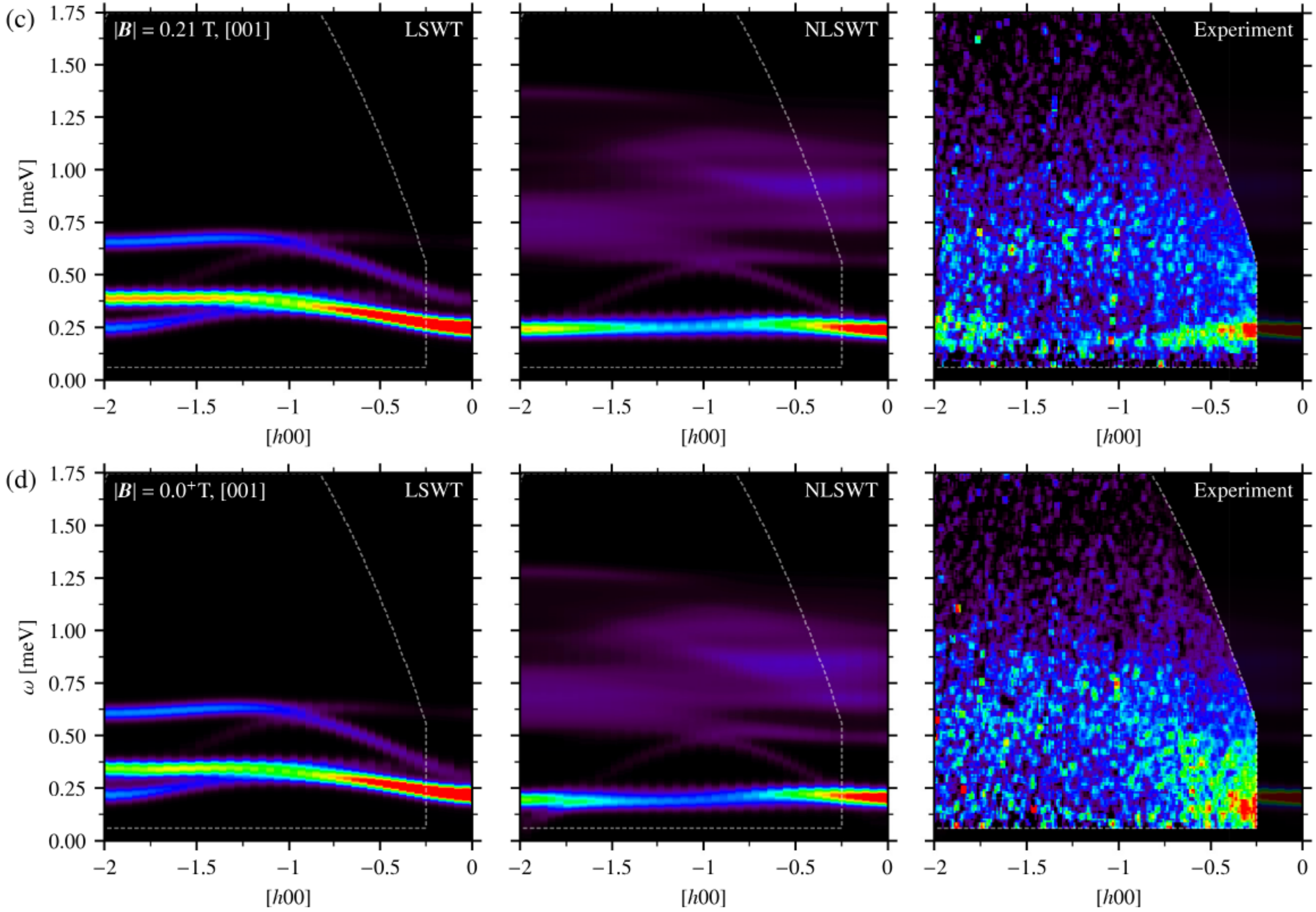


Magnons in spin-ladder

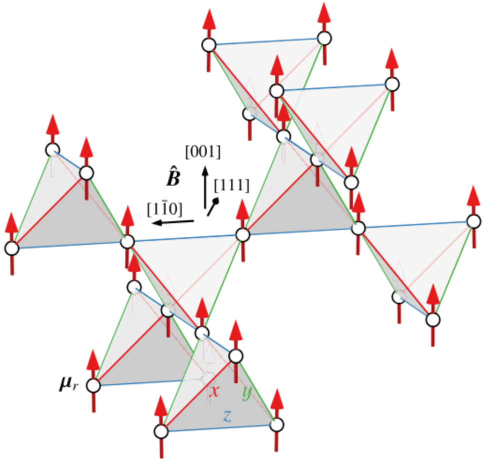
Triplons in Dimerized magnet



Beauvois et al, PRB (2018); Stone et al, Nature (2005), Plumb et al, Nat Phys (2016)



$Yb_2Ti_2O_7$ (anisotropic FM)



- … often amount of (apparent) decay isn't *enough* as compared to experiment

Rau *et al*, Phys. Rev. B **100**, 104423 (2019)

Pseudo-Goldstone Modes

Pseudo-Goldstone Modes

- Rotation of spins in "soft" direction is *shift of bosons*

$$a_r \to a_r + \lambda_r$$

Determined by ordering pattern & direction of deformation

- Accidental degeneracy implies **zero mode**

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^\dagger & \boldsymbol{A}^\intercal \end{pmatrix} \begin{pmatrix} \lambda \\ \lambda^\dagger \end{pmatrix} = 0$$

Two kinds of zero modes

– **Type I:** Conjugate momentum is "hard"

– **Type II:** Conjugate momentum also soft

i.e. $H \sim P^2$

i.e. $H = 0$

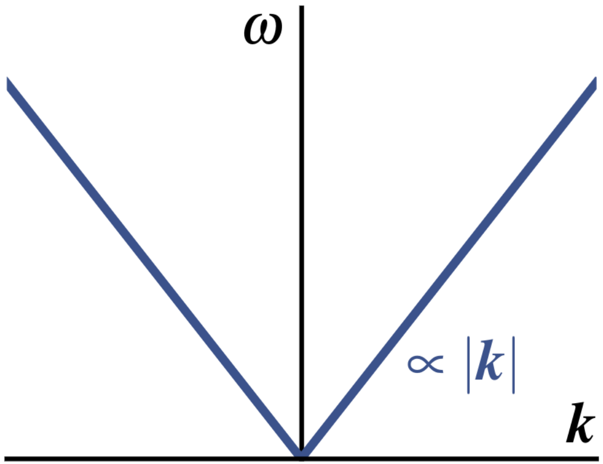
- More mathematically for a mode at $\boldsymbol{k} = 0$:

$$\begin{pmatrix} A_{\boldsymbol{k}} & B_{\boldsymbol{k}} \\ -B^*_{\boldsymbol{k}} & -A_{\boldsymbol{k}} \end{pmatrix} \xrightarrow{\text{at } \boldsymbol{k}=0} \overset{\textit{Type I}}{\begin{pmatrix} A_{\boldsymbol{0}} & A_{\boldsymbol{0}} \\ -A_{\boldsymbol{0}} & -A_{\boldsymbol{0}} \end{pmatrix}} \quad \textit{or} \quad \overset{\textit{Type II}}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$$

Non-diagonalizable (Jordan block)

Type I

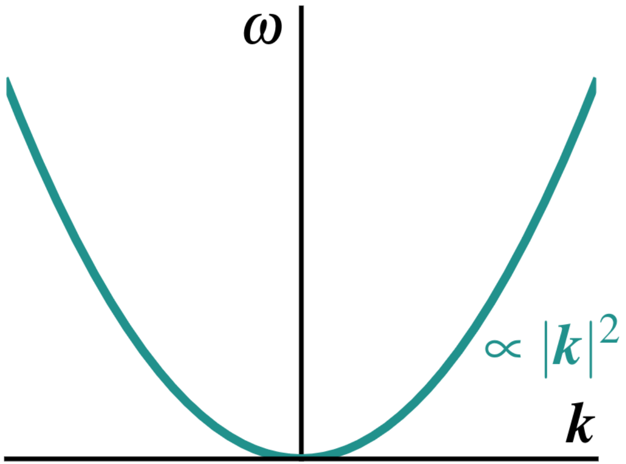
e.g. AFM



Perturbations to zero mode ~ ε^{1/2}

Type II

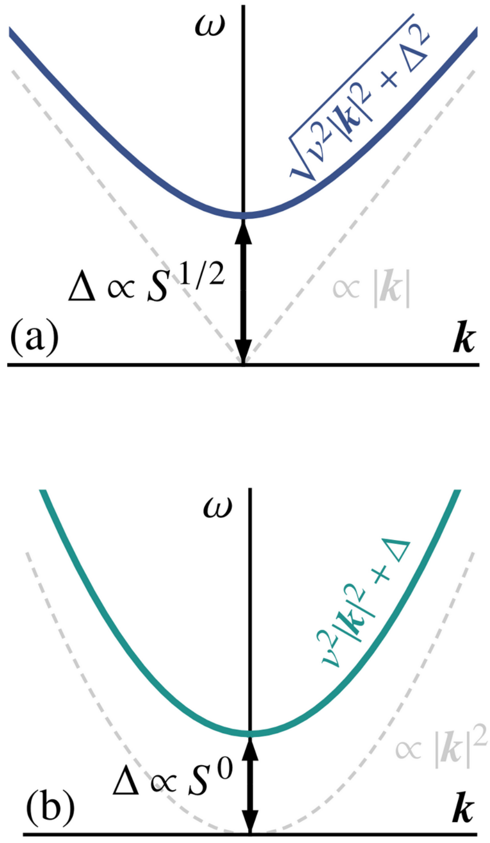
e.g. FM



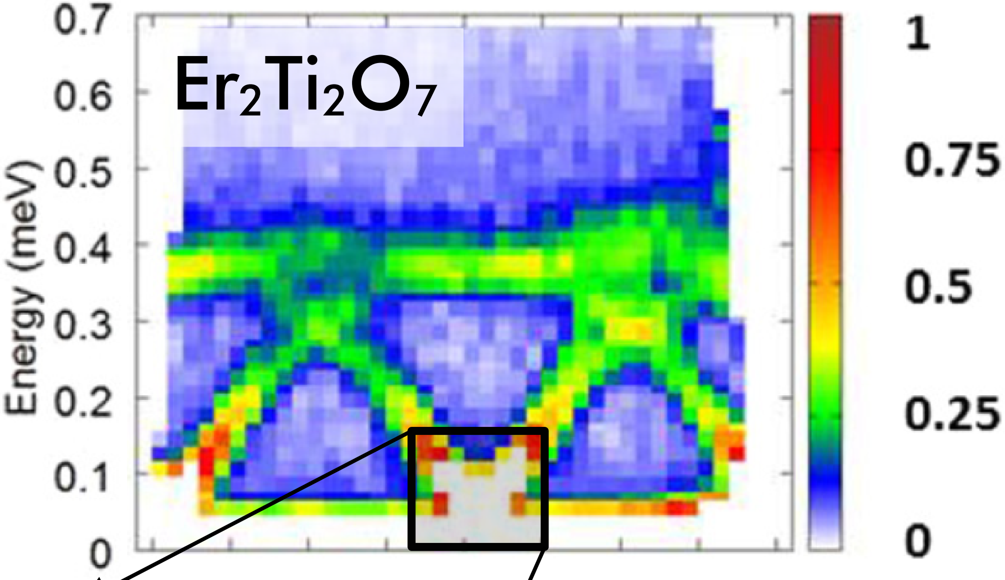
Perturbations to zero mode ~ ε

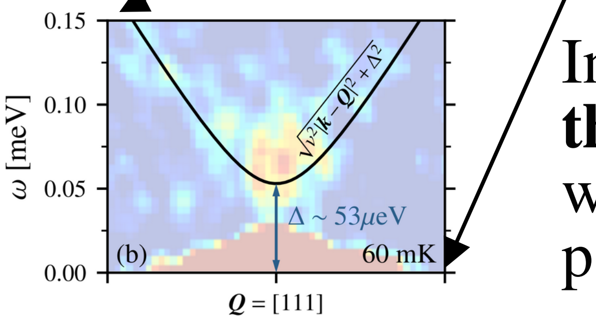
Pseudo-Goldstone gaps

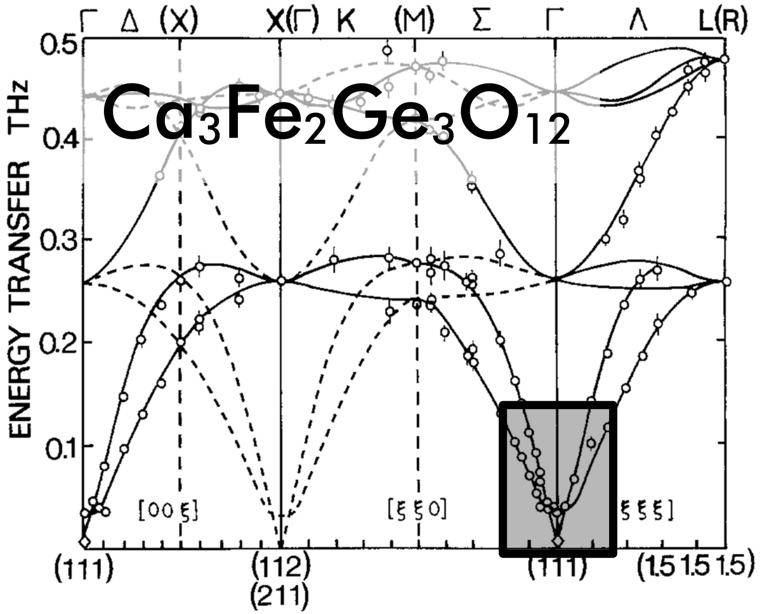
- Generated by *quantum fluctuations*
- Can be calculated at O($1/S^2$) in spin-wave theory
  - Directly via perturbation theory in magnon-magnon interactions
  - Indirectly via **curvature formula*** involving fluctuation corrected energy density
- **Intrinsic interaction effect**
- Expect *(hope)* they are small, perturbative in $1/S$



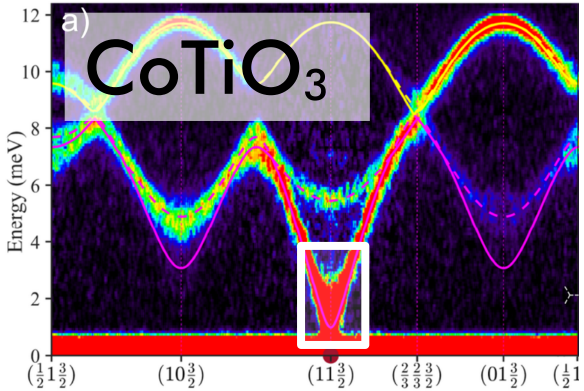
*Rau *et al*, Phys. Rev. Lett **121**, 237201 (2018)







In *all* cases less than **~10% of the spin-wave bandwidth,** weak effects away from pseudo-Goldstone mode

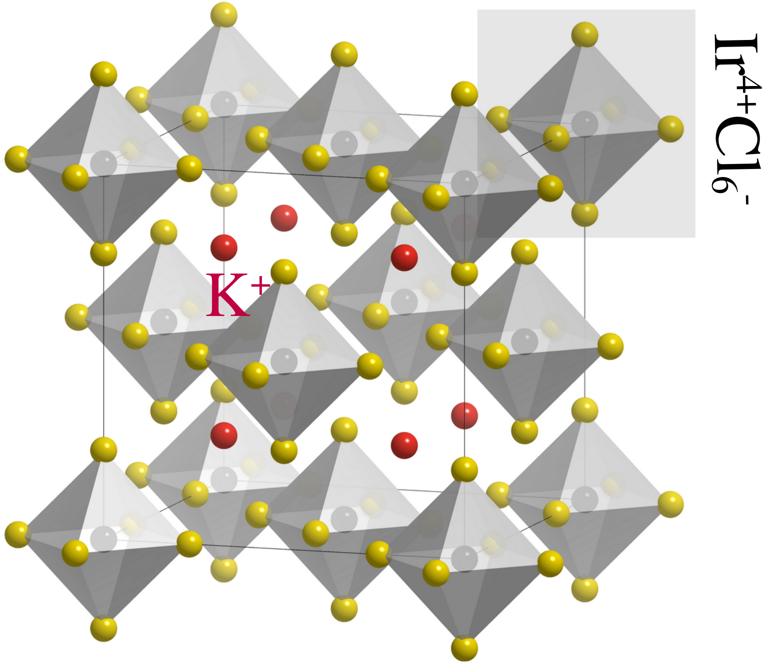


Ross *et al*, Phys. Rev. Lett. **112**, 057201 (2014), Petit *et al*, Phys. Rev. B **90**, 060410(R) (2014); T. Bruekel *et al*, Z. Phys. B **72**, 477 (1988), Elliot *et al*, Nat. Comm. **12** 3936 (2021), ….

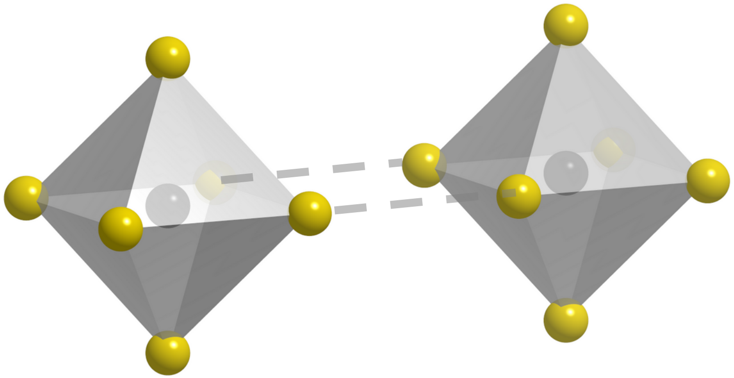
K_2IrCl_6

- Cubic crystal (SG #225)
- Magnetic Ir^{4+} ions ($5d^5$)
  - Octahedral Cl^- cages
  - *Strong spin-orbit coupling*
  - $\boldsymbol{J_{\mathrm{eff}} = 1/2}$ **doublets**
- **Face-centered cubic lattice** (no detectable distortion)
- Same bond *symmetry* as in (idealized) Kitaev materials

No trigonal or tetragonal distortions



Separated; not edge-shared



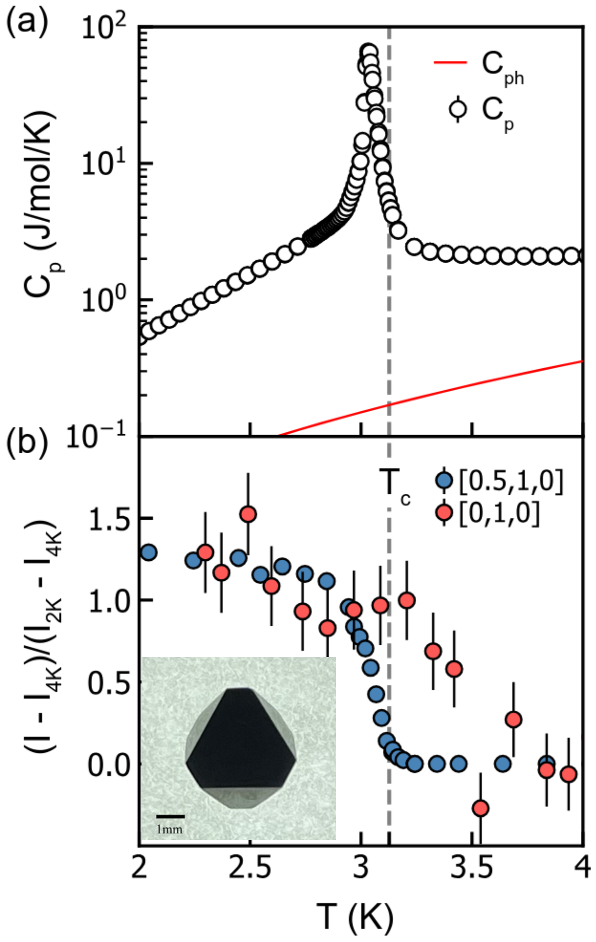
Cooke *et al*. Proc. Royal Soc. A 250, 97–109 (1959), *ibid*, 84–96, … ; Khan *et al*, Phys. Rev. B. **99**, 144425 (2019)

One sharp peak (no shoulders)

Magnetic Order

- Order appears at low temperature T_N ~ 3 K via sharp phase transition
  - Curie-Weiss temperature estimated to be Θ_{CW} ~ 30 K
  - **Highly frustrated**
- Bragg peaks appear at [½,1,0] and equivalents below T_N
  - *Antiferromagnetic "Type III" order*
- Polarization analysis yields moment *parallel* to ½ direction

Looks continuous

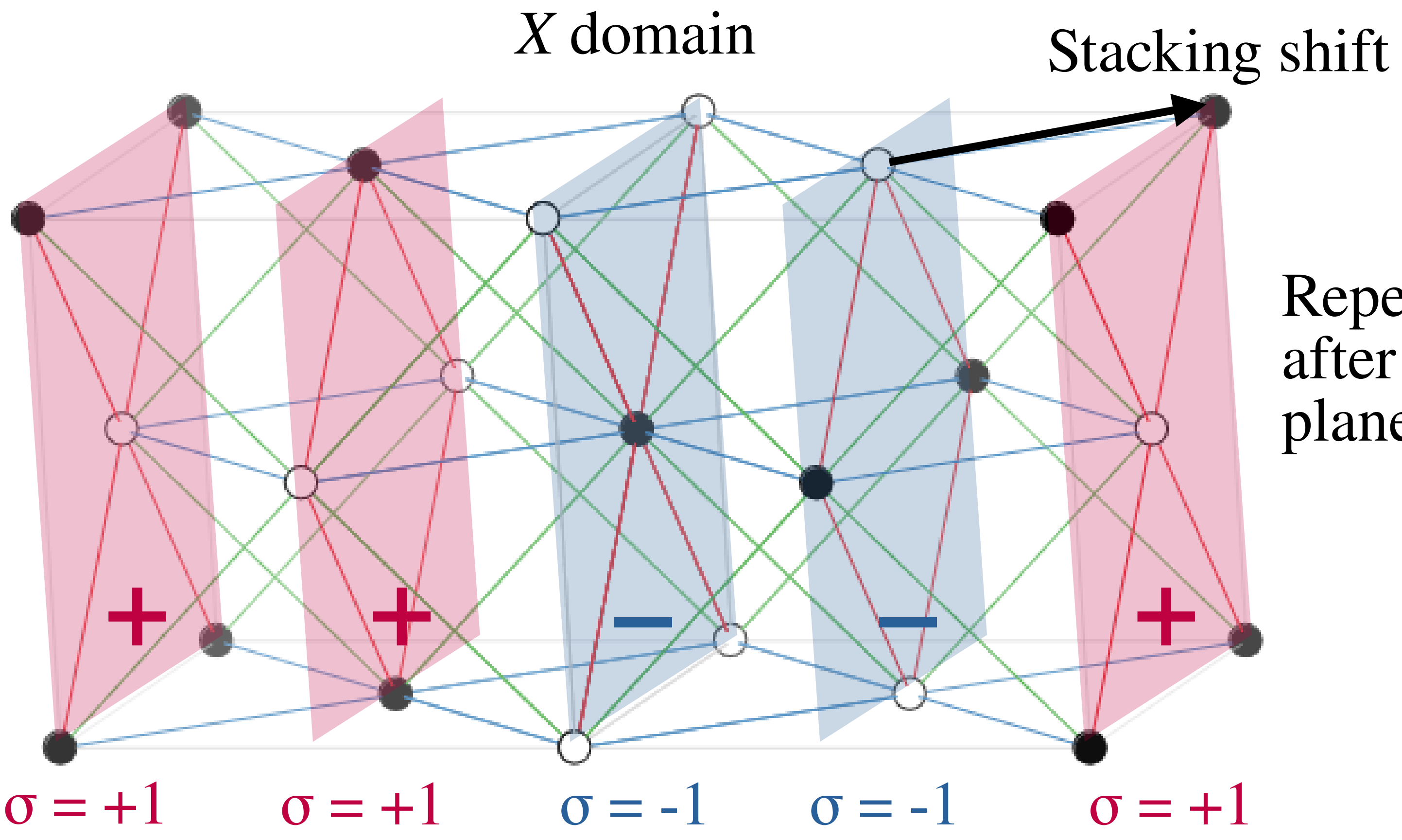


Hutchings *et al*, Proc. Phys. Soc. **91**, 928 (1967),
Lynn *et al.*, Phys. Rev. Lett. **37**, 154 (1976)

Wang *et al.*, Phys. Rev. X **15**, 021021 (2025)

Type-III: AFM planes stacked ++−−...

Each plane is AFM aligned



Repeats after 4 planes

Model

- Anisotropic spin-1/2 exchange model on FCC lattice

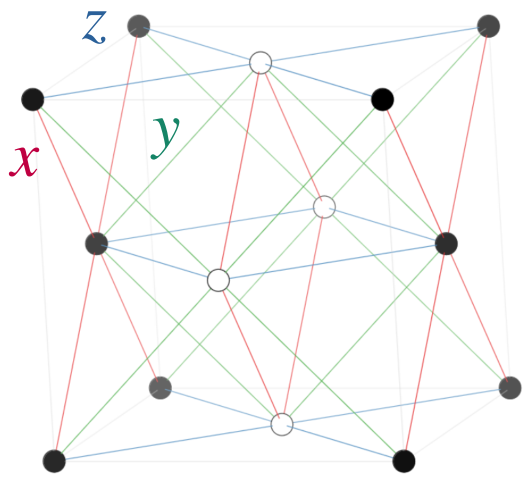
$$H = \sum_{\langle ij\rangle_{\alpha\beta(\gamma)}} \left[J\boldsymbol{S}_i \cdot \boldsymbol{S}_j + K S_i^\gamma S_j^\gamma + (-1)^{\sigma_{ij}^{\alpha\beta}} \Gamma \left(S_i^\alpha S_j^\beta + S_i^\alpha S_j^\beta \right) \right] + J_2 \sum_{\langle ij\rangle_2} \boldsymbol{S}_i \cdot \boldsymbol{S}_j$$

- Three bond types: x, y, z. $\alpha\beta(\gamma)$ is cyclic permutation
- Γ has sign structure determined by

$$\sigma_{ij}^{\alpha\beta} \equiv \mathrm{sgn}(d_{ij}^\alpha d_{ij}^\beta)$$

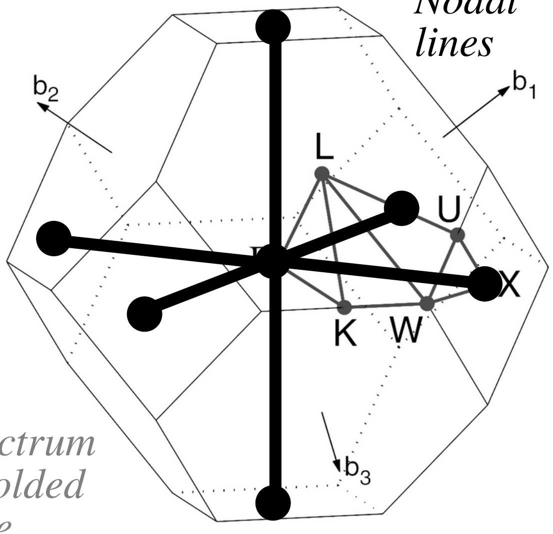
where $\boldsymbol{d}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$ is the bond direction

- Isotropic J_2 for simplicity



Judd *et al*. Proc. Royal Soc. A **250**, 110-120 (1959); Cook *et al*, Phys. Rev. B **92**, 020417(R) (2015), Khan *et al*, Phys. Rev. B. **99**, 144425 (2019), Bhaskaran *et al*., Phys. Rev. B **104**, 184404 (2021), ...

Ordered Phase Excitations

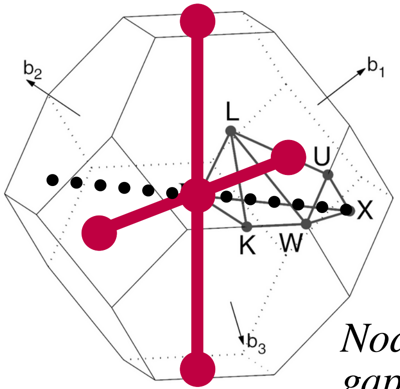


Nodal lines

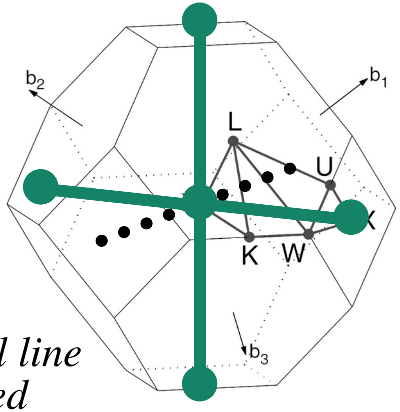
Spectrum in folded zone

- **Three nodal lines** in *all* ground states in Heisenberg limit
- Correspond to continuous mixing with other stacked states – "**pseudo-Goldstone modes**"

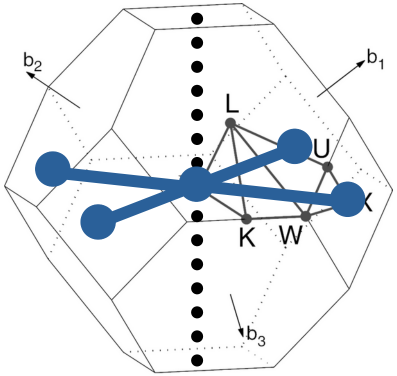
X stacked



Y stacked

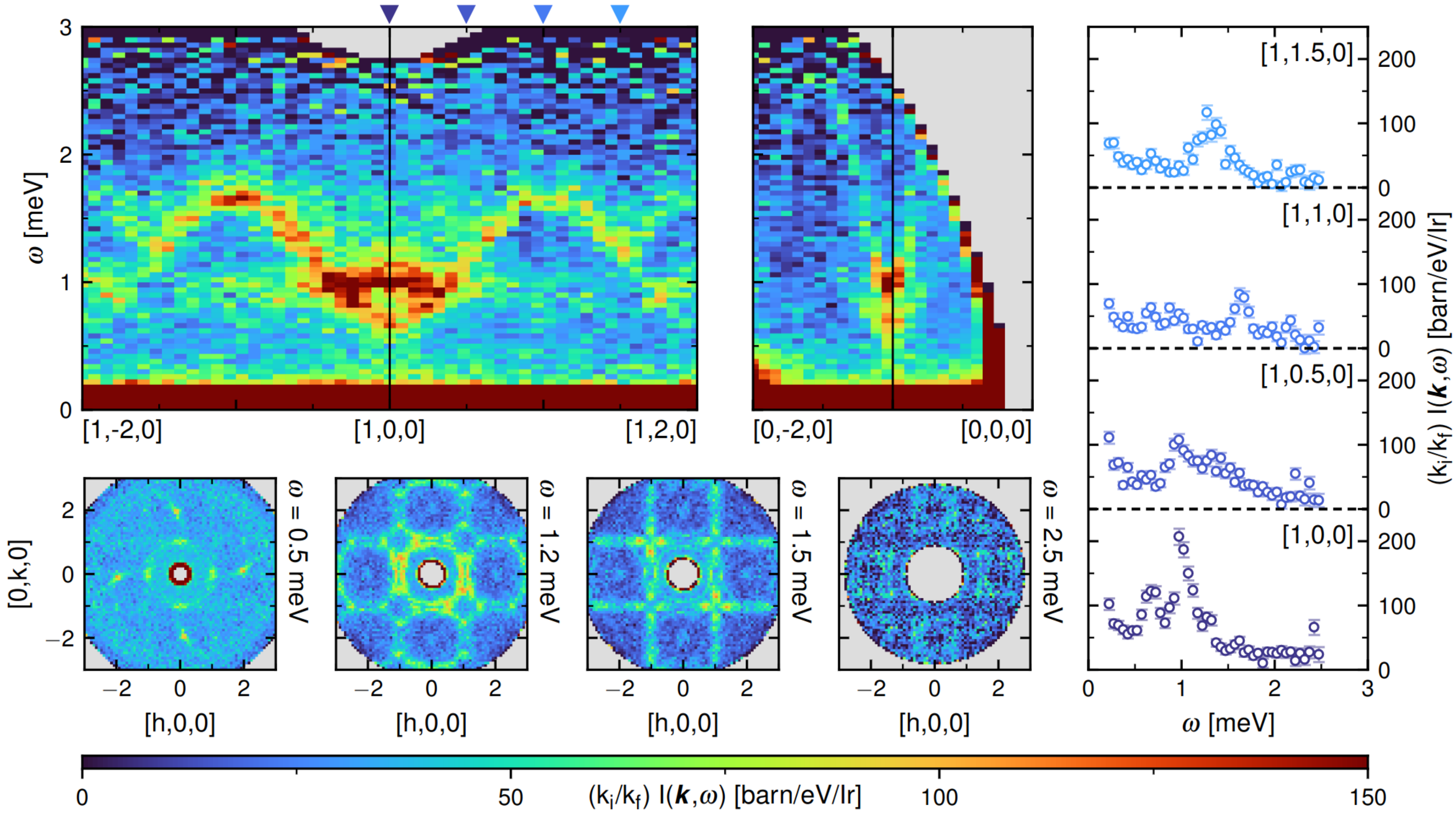


Z stacked



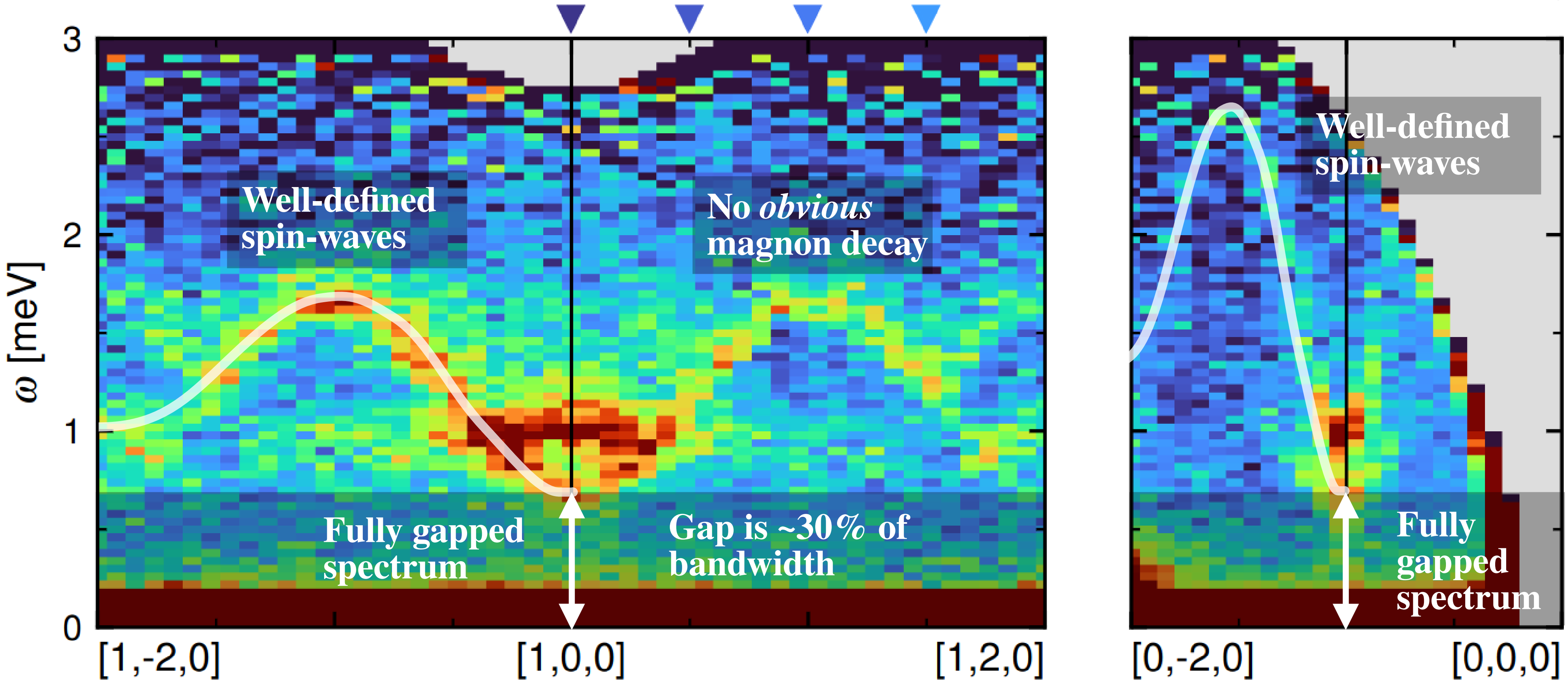
Nodal line gapped

- Nodal line **along stacking direction is lifted** by Kitaev interaction
- **Two nodal lines left**



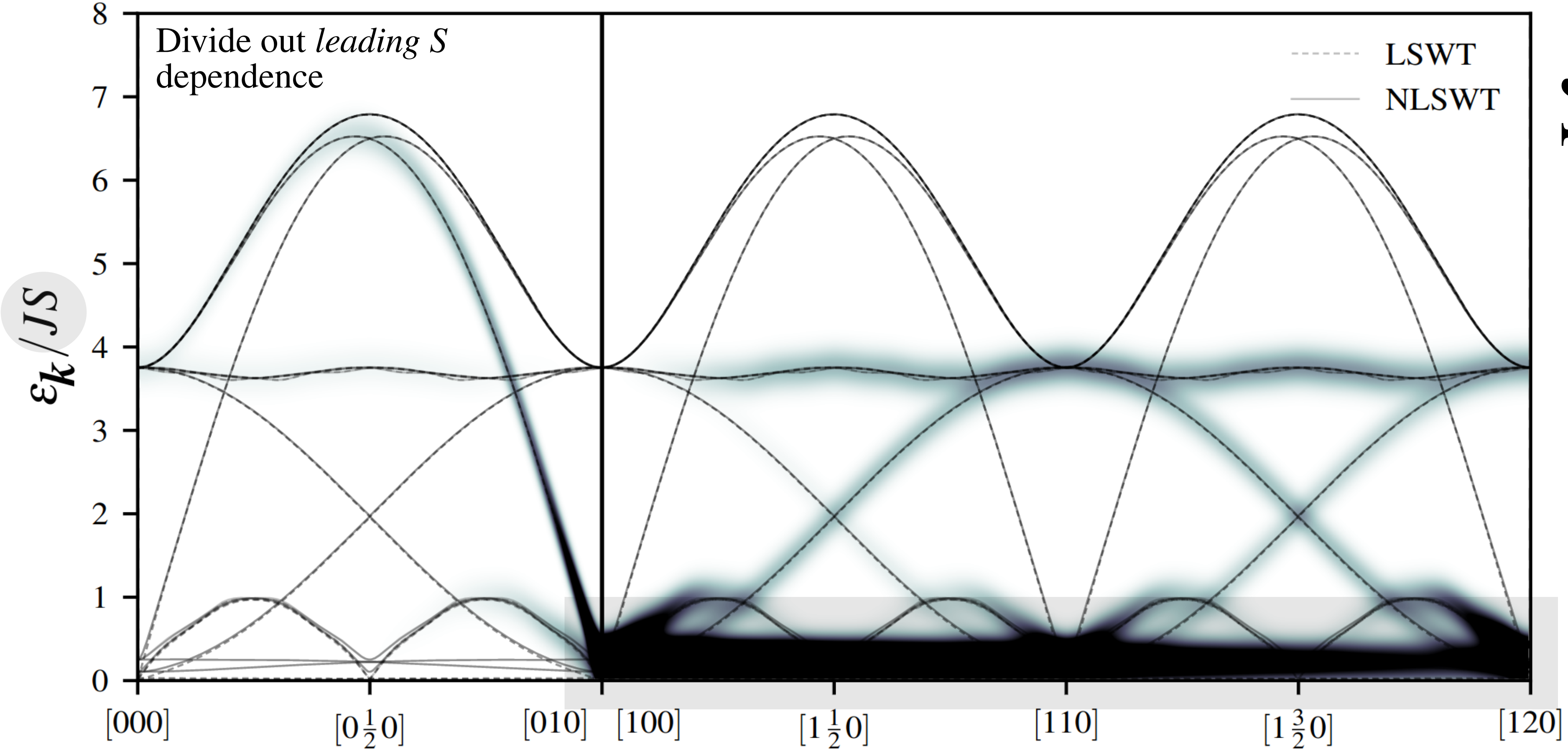
*Iridium is **strong neutron absorber**

Wang *et al.*, Phys. Rev. X **15**, 021021 (2025)



Very different than expectations!

Wang *et al.*, Phys. Rev. X **15**, 021021 (2025)

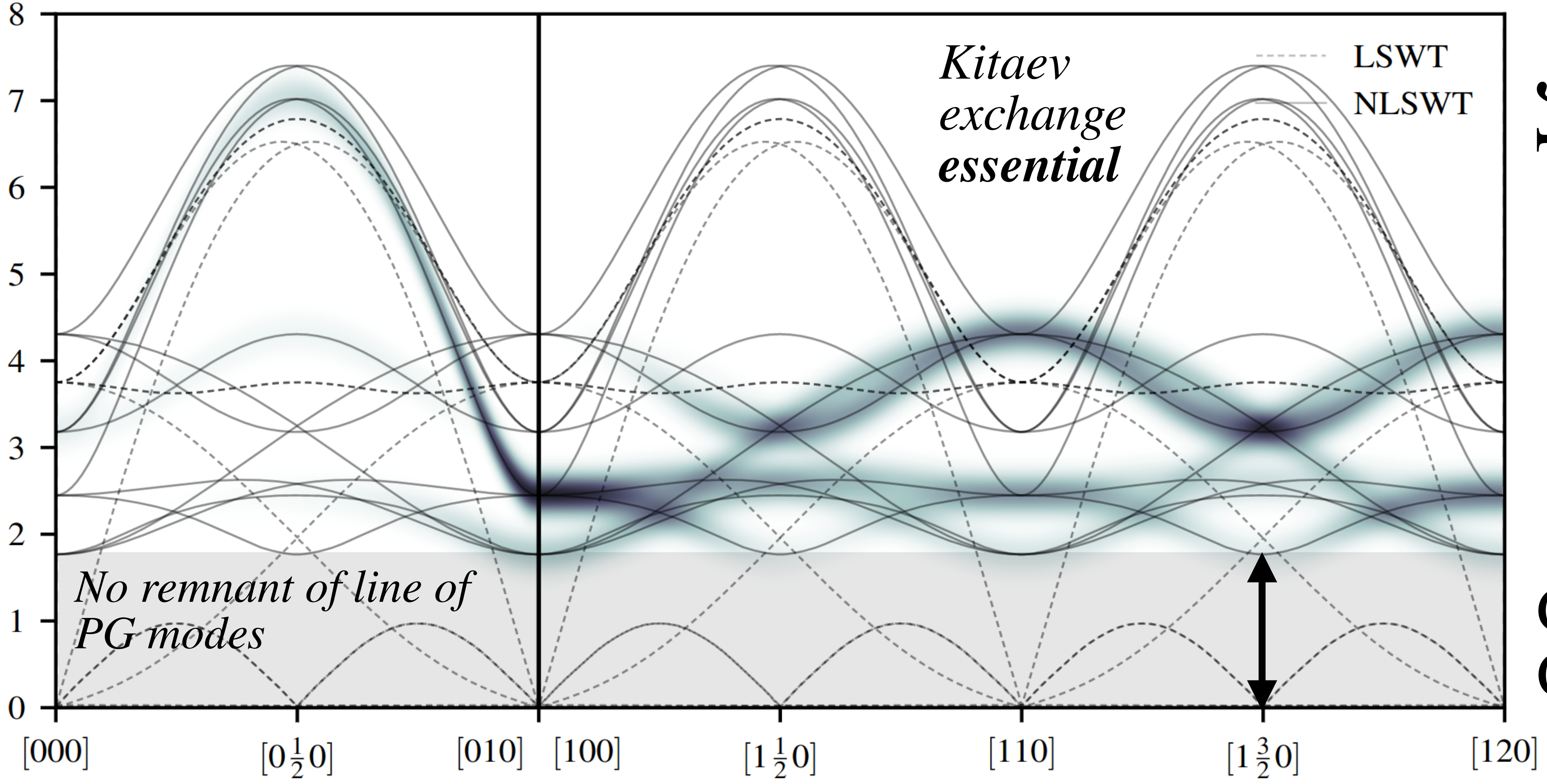


Type III

Line of zero modes

$K/J = 0.20,\ 1/S \to 0$

LSWT



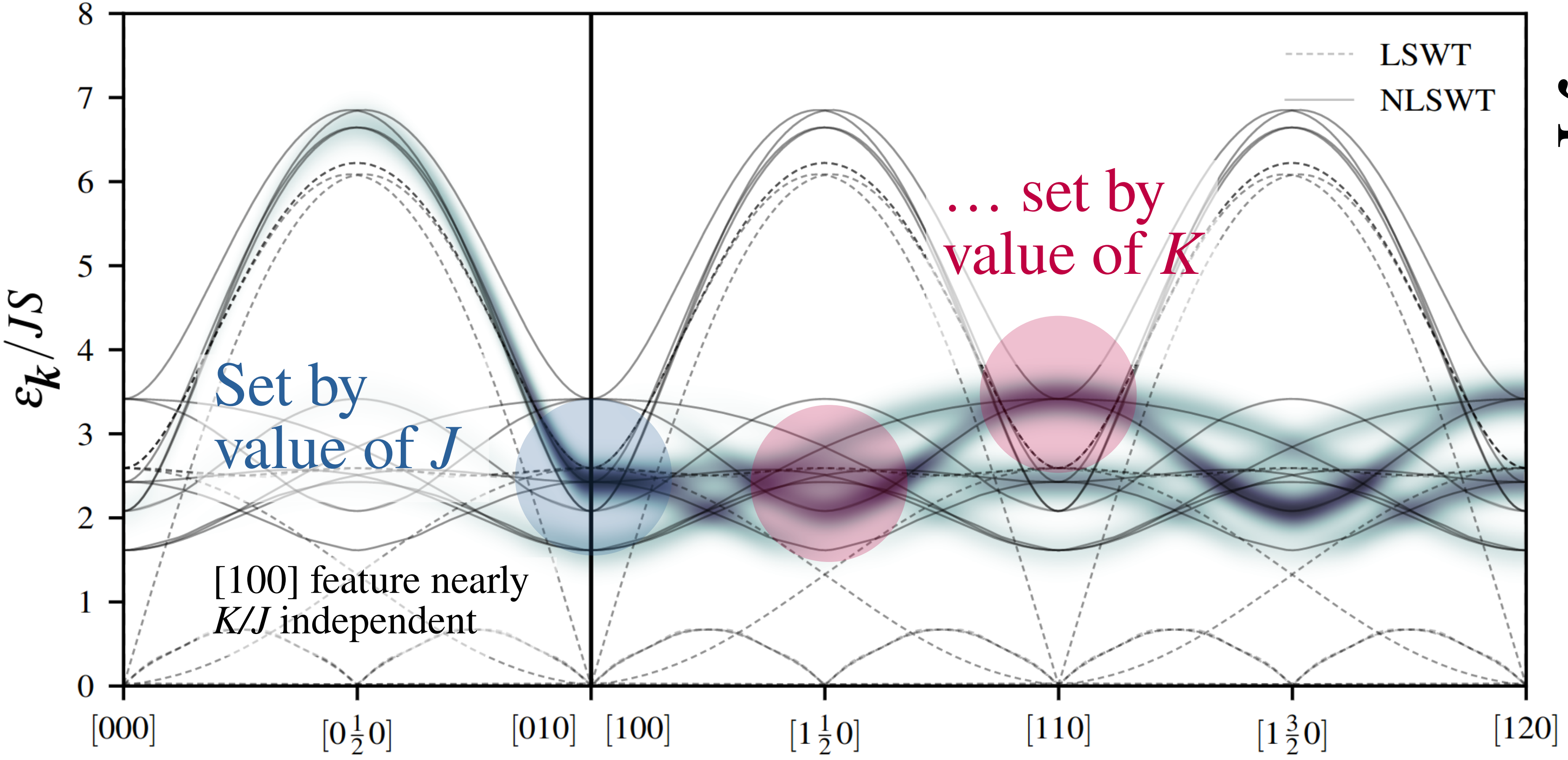
Type III

Gap is O(J)

$K/J = 0.20$, $S = 1/2$

NLSWT

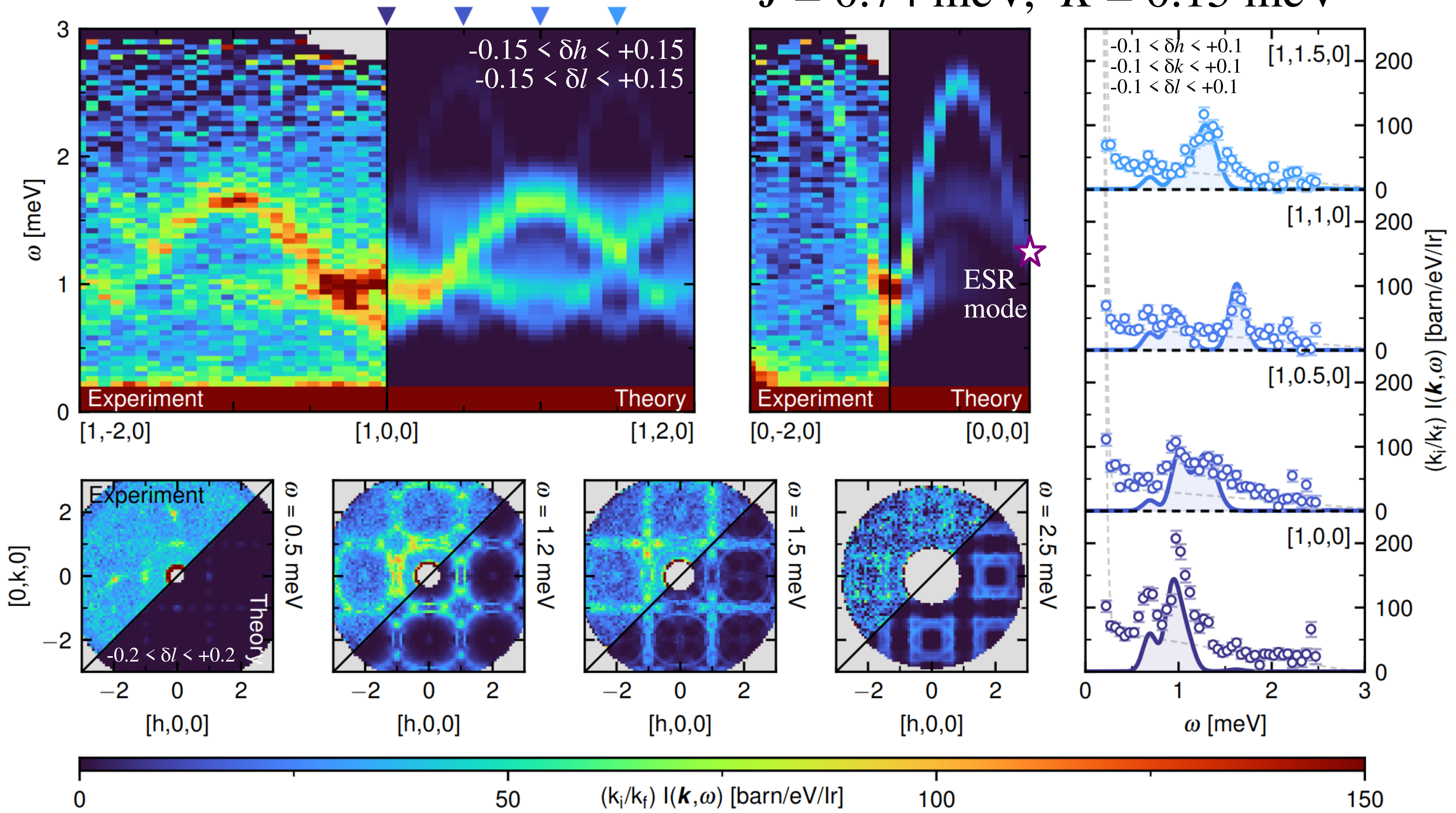
Type III



Set $\Gamma = 0$ (no decay) and $J_2 = 0$ (simplicity)

Including: Experimental averaging, Ir^{4+} form factor

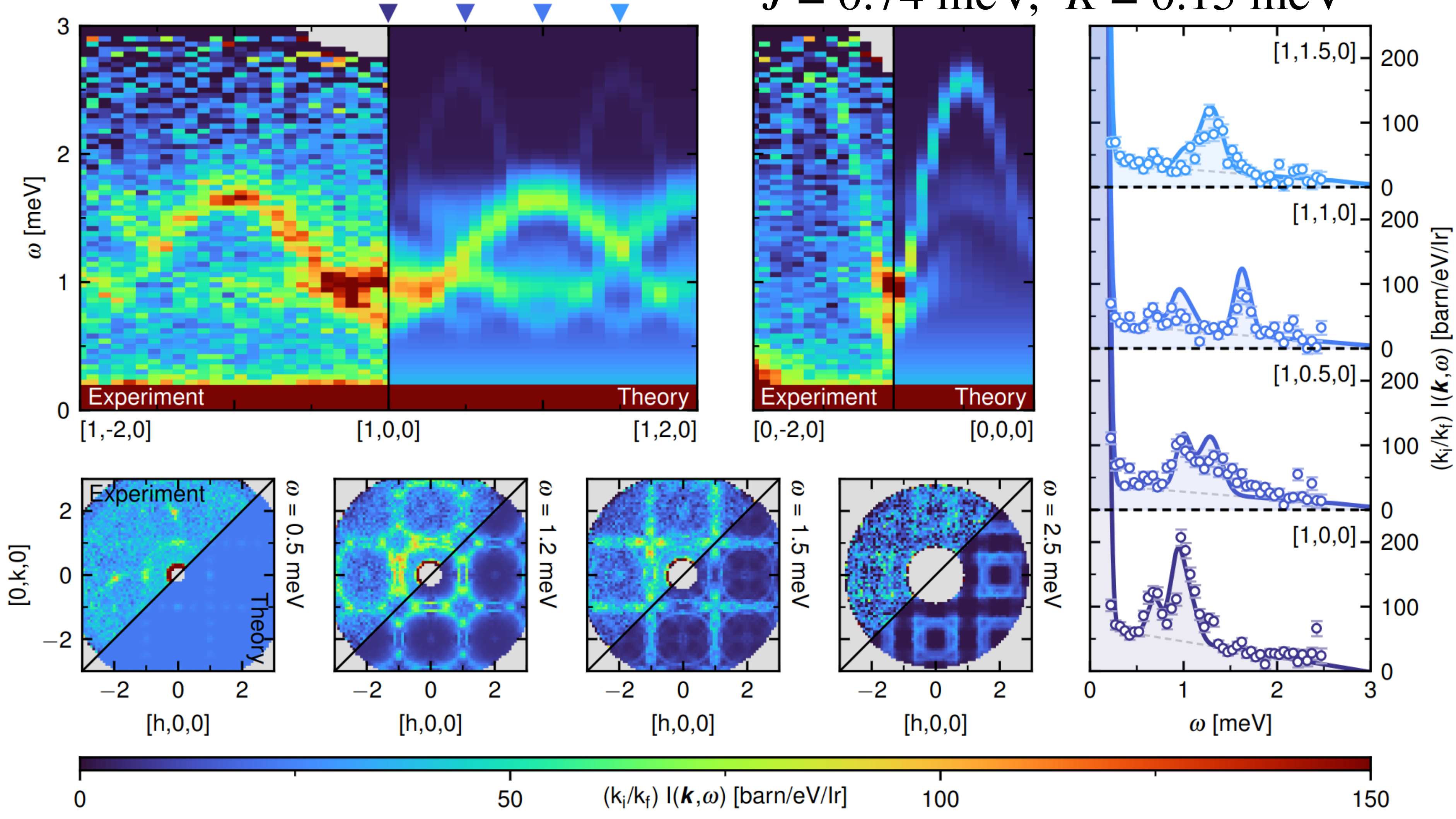
$J = 0.74$ meV, $K = 0.15$ meV



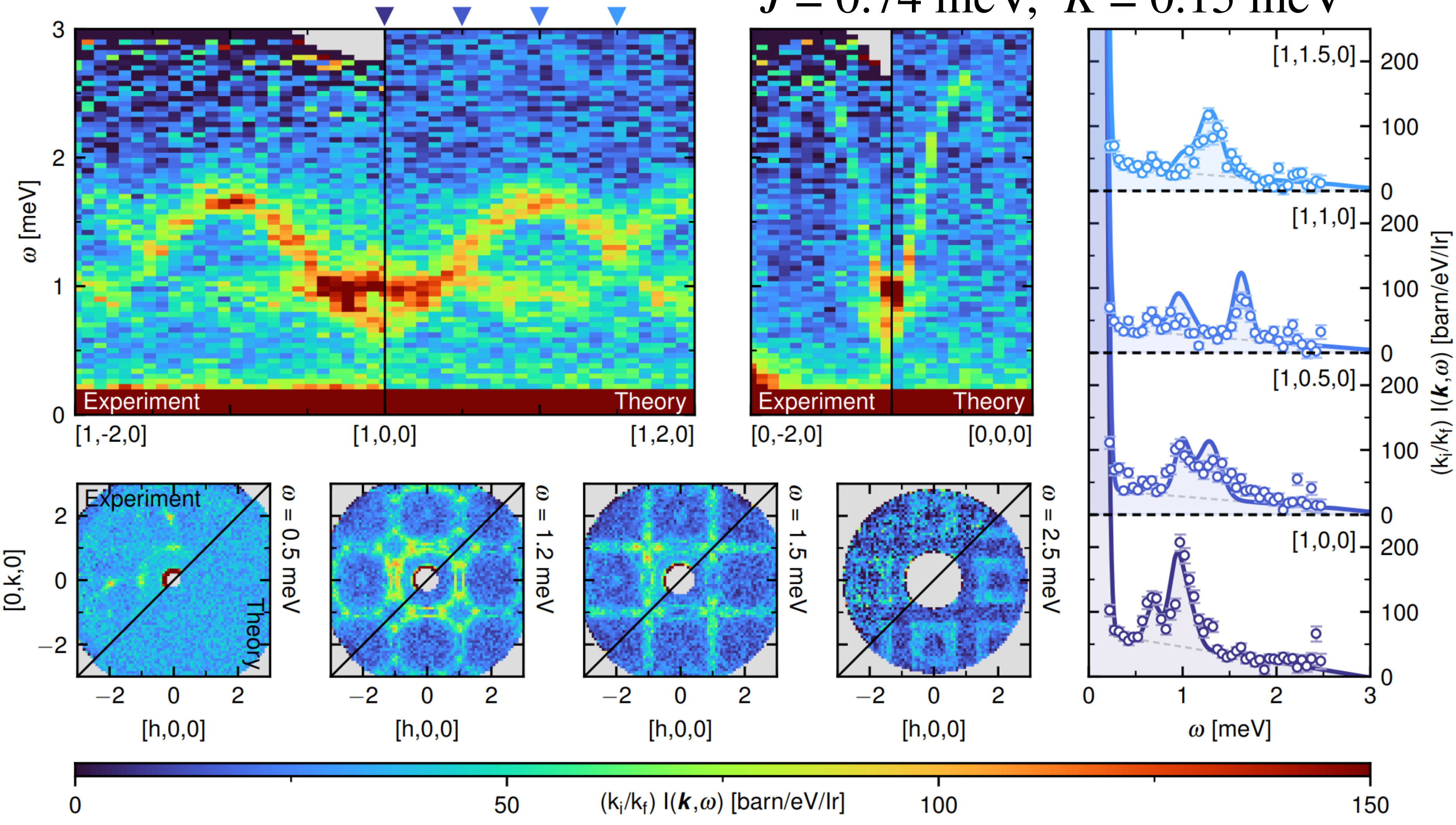
*Also gives Θ_{CW} ~ 27.5 K and mode at 1.22 meV ($\boldsymbol{q} = 0$) mode from ESR [Bhaskaran *et al.*, Phys. Rev. B **104,** 184404 (2021)]

Including: Experimental averaging, Ir^{4+} form factor + *background*

$J = 0.74$ meV, $K = 0.15$ meV



$J = 0.74$ meV, $K = 0.15$ meV



Including: Experimental averaging, Ir^{4+} form factor + background + *noise*

Conclusion

- Spin-wave theory can describe the excitations of ordered states of quantum magnets
- Often linear spin-waves work surprising well even at $S = 1/2$

… but magnon-magnon interactions can often be important

- (More) quantitative comparison to small S limit (& experiment)
- Magnons are not always sharp; spontaneous decay can imbue them a finite lifetime
- Other features also *require* interactions to describe at any level: pseudo-Goldstone modes, transport, ...

Self-consistent mean-field theory

- Corrections are *static* (no frequency dependence)
- Can be thought of as modifications of linear spin wave dispersion

$$\delta A_{\boldsymbol{k}} = \frac{1}{N} \sum_{\boldsymbol{q}} \left[V_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{0}} \langle a^{\dagger}_{\boldsymbol{q}} a_{\boldsymbol{q}} \rangle + \frac{1}{2} \left(D_{\boldsymbol{q},-\boldsymbol{q},\boldsymbol{k}} \langle a^{\dagger}_{\boldsymbol{q}} a^{\dagger}_{-\boldsymbol{q}} \rangle + \text{c.c.} \right) \right]$$

$$\delta B_{\boldsymbol{k}} = \frac{1}{N} \sum_{\boldsymbol{q}} \left[D_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}} \langle a^{\dagger}_{\boldsymbol{q}} a_{\boldsymbol{q}} \rangle + \frac{1}{2} V_{\boldsymbol{q},-\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q}} \langle a_{\boldsymbol{q}} a_{-\boldsymbol{q}} \rangle \right]$$

- *Alternatively:* from a Hartree-Fock like “decoupling” of the four-boson interactions into all hopping and pairing channels

Self-consistent mean-field theory

- Add and subtract corrections to quadratic part

$$\delta H_2 = \sum_{\boldsymbol{k}} \left[\delta A_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{1}{2!} \left(\delta B_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a^\dagger_{-\boldsymbol{k}} + \text{h.c.} \right) \right]$$

- Treat added part with LSWT and subtracted part perturbatively

$$H_2 + H_4 = (H_2 + \delta H_2) + (H_4 - \delta H_2)$$

Modifies LSWT energies, eigenvectors

Perturbative contribution depends on $\delta A_k, \delta B_k$

Self-consistent mean-field theory

- *Choose* the correction terms so that they cancel the interaction corrections

Static correction from H_4

$$\boxed{\Sigma_{\mathrm{s}}^{-+}(\boldsymbol{k})} - \delta A_{\boldsymbol{k}} = 0,$$

$$\boxed{\Sigma_{\mathrm{s}}^{--}(\boldsymbol{k})} - \delta B_{\boldsymbol{k}} = 0.$$

Depends on value of δA_k, δB_k

- *Practically:* Self-consistent interation for the correction terms

$$\delta A_{\boldsymbol{k}}^{(n+1)} = \Sigma_{\mathrm{s}}^{-+}(\boldsymbol{k})\Big|_{\delta A^{(n)},\delta B^{(n)}}$$

$$\delta B_{\boldsymbol{k}}^{(n+1)} = \Sigma_{\mathrm{s}}^{--}(\boldsymbol{k})\Big|_{\delta A^{(n)},\delta B^{(n)}}$$

Iterate until maximum error falls below threshold ε