Universal temperature-dependent power law excitation gaps in frustrated quantum spin systems harboring order-by-disorder

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(Dated: May 30, 2025)

When magnetic moments are subject to competing or frustrated interactions, continuous degeneracies that are not protected by any symmetry of the parent Hamiltonian can emerge at the classical (mean-field) level. Such "accidental" degeneracies are often lifted by both thermal and quantum fluctuations via a mechanism known as order-by-disorder (ObD). The leading proposal to detect and characterize ObD in real materials, in a way that quantitatively distinguishes it from standard energetic selection, is to measure a small fluctuation-induced *pseudo-Goldstone gap* in the excitation spectrum. While the properties of this gap are known to leading order in the spin wave interactions, in both the zero-temperature and classical limits, the pseudo-Goldstone (PG) gap in quantum magnets at finite temperature has yet to be characterized. Using non-linear spin wave theory, we compute the PG gap to leading order in a 1/S expansion at low temperature for a variety of frustrated quantum spin systems. We also develop a formalism to calculate the PG gap in a way that *solely* uses linear spin-wave theory, circumventing the need to carry out tedious quantum many-body calculations. We argue that, at leading order, the PG gap acquires a distinct power-law temperature dependence, proportional to either T^{d+1} or $T^{d/2+1}$ depending on the gapless dispersion of the PG mode predicted at the mean-field level. Finally, we examine the implications of these results for the pyrochlore oxide compound $Er_2Ti_2O_7$, for which there is compelling evidence of ObD giving rise to the experimentally observed long-range order.

In systems of interacting degrees of freedom, both thermal and quantum fluctuations typically act to destabilize broken symmetries and long-range order. In certain condensed matter systems, these fluctuations can be so pronounced that they completely suppress order even at absolute zero temperature. A striking example is liquid helium, which avoids freezing into a crystalline solid at atmospheric pressure due to strong quantum zero-point fluctuations [1]. Within the field of magnetism, the quest for quantum spin liquids-states that lack long-range magnetic order down to the lowest temperatures-has long been a central endeavor [2-6]. One preeminent setting for this search considers highly frustrated magnetic systems where competing spin-spin interactions produce an exponentially large number of classically degenerate ground states [7, 8]. This degeneracy leads to an extensive entropy that can prevent magnetic ordering down to absolute zero temperature, even in three-dimensional systems, giving rise to a classical spin liquid [8–10]. Such a classical perspective has motivated an intense theoretical and experimental exploration of quantum analogs of such systems.

An intriguing intermediate situation between energetically stabilized long-range magnetic order and fluctuation-driven spin liquid states happens in certain models of frustrated spin systems that possess a sub-exponentially large (in the system size) manifold of classical ground states. In these systems, the *accidental* degeneracies are not protected by global symmetries of the spin Hamiltonian but rather emerge from the specific form of the spin-spin interactions [11]. Although no unique classical ground state (modulo global symmetries) is energetically favored at the mean-field level, thermal or quantum fluctuations can lift the degeneracy, stabilizing a subset of configurations within the manifold of classical ground states and cause long-range magnetic order—this is the celebrated



FIG. 1. Schematic picture depicting the spectrum of a PG modes for (a) type I with a linear dispersing mode, $\epsilon = v|\mathbf{k}|$, and (b) type II. with a quadratic dispersing mode, $\epsilon = v|\mathbf{k}|^2$. In both cases, the mode acquires a gap Δ via the ObD mechanism, modifying the dispersion. Quantum fluctuations at zero temperature (T = 0) generates a gap Δ_0 (quantum ObD), which then *increases* with T as a power-law, T^{d+1} and $T^{d/2+1}$, for type I and type II mode, respectively.

phenomenon of *order-by-disorder* (ObD) [12–14], a conceptual cornerstone of frustrated magnetism.

While ObD driven by thermal or quantum fluctuations has been well-established in an abundant number of theoretical models of frustrated spin systems [12–48], a long-standing question is whether ObD is realized in real magnetic materials and, if so, what experimental signatures can unambiguously demonstrate its role in stabilizing the observed long-range order. The conventional approach to experimentally confirming ObD has been rather indirect, proceeding by constructing a theoretical model of the material, demonstrating ObD within that framework, and then experimentally verifying a maximum number of the model's predictions against the material properties [39, 49–53]. This program for identifying ObD in real materials raises two important concerns. Firstly, it presents significant methodological challenges, as establishing detailed quantitative agreement between theory and experiment where ObD is operating necessitates intricate high-order quantum many-body perturbation theory calculations. Secondly, even in cases where it can be successfully executed, the above program leaves one wanting for a deeper understanding of ObD, such as asking whether there may exist more direct experimental evidence for ObD in a material that do not rely on a back and forth "dialogue" between theoretical calculations and the fitting of experimental data.

Two recent developments [43, 54] have begun addressing the above two issues by considering the formation of an ObD-induced energy gap Δ in the otherwise gapless pseudo-Goldstone (PG) mode (i.e. spin-wave excitations) predicted by mean-field theory (see Fig. 1). The first development is the observation that, for a given Hamiltonian model for a quantum spin-S system, the curvatures of classical plus quantum zeropoint energies computed at O(S) are sufficient to determine the PG zero-temperature gap Δ_0 exactly to $O(S^0)$, without the necessity of performing involved self-energy calculations due to magnon-magnon interactions [43]. The second development is the observation that frustrated classical spin systems, displaying thermal but no quantum ObD, feature a temperaturedependent gap generated by magnon interactions scaling like $\Delta(T) \propto T^{\mu}$ with $\mu = \frac{1}{2}$ ($\mu = 1$) for linearly dispersing (quadratically dispersing) PG modes [54]. These two results beg for two important follow-up questions: "What is the expected temperature-dependence of $\Delta(T)$ for a *quantum* treatment of models with linearly or quadratically dispersing PG modes?" and "Can one compute such temperature dependence through a generalization of the framework of Ref. [43] involving curvatures of the free energy density using only linear spin-wave theory?" In this paper, we answer the second question in the affirmative, and show that the above exponent μ become d + 1 (linearly dispersing PG mode, $\epsilon_k \propto |\mathbf{k}|$) and d/2 + 1(quadratically dispersing PG mode, $\epsilon_k \propto |k|^2$), respectively (see Fig. 1). In short, we have (i) identified a "smoking-gun" experimental evidence for ObD in a spin system in the form of two "universal" types of ObD-generated temperature dependent gaps $\Delta(T)$ to the PG mode, and (*ii*) propose a straightforward methodology to compute $\Delta(T)$ that is unburdened from complex finite-temperature quantum-many body calculations.

Spin-wave theory—To investigate the thermal properties of the fluctuation-induced gap, we use the formalism of spin-wave theory, extending the calculations of Ref. [43] to finite temperature to incorporate thermal fluctuation effects. Formally, spin operators are mapped to Holstein-Primakoff bosons [55] and subsequently expanded in powers of 1/S, where S is the spin quantum number [56]. Substituting this expansion into the Hamiltonian and keeping only the bilinear terms results in the linear spin-wave Hamiltonian

$$H_2 = \sum_{\boldsymbol{k}} \sum_{\alpha \alpha'} \left[A_{\boldsymbol{k}}^{\alpha \alpha'} a_{\boldsymbol{k},\alpha}^{\dagger} a_{\boldsymbol{k},\alpha'} + \frac{1}{2} \left(B_{\boldsymbol{k}}^{\alpha \alpha'} a_{\boldsymbol{k},\alpha}^{\dagger} a_{-\boldsymbol{k},\alpha'}^{\dagger} + \text{H.c.} \right) \right],$$
(1)

where $a_{k,\alpha}$ is a Holstein-Primakoff boson with wavevector k on sublattice α of the magnetic unit cell, and the coefficients $A_k^{\alpha\alpha'}$ and $B_k^{\alpha\alpha'}$ depend on both the magnetic exchange interactions and the specific classical ground state configuration(s) considered [56]. The quasiparticle dispersion is then determined by the eigenvalues of the (non-Hermitian) Bogoliubov-de Gennes matrix [57]

$$\sigma_3 M_k = \sigma_3 \begin{pmatrix} A_k & B_k \\ B_k^{\dagger} & A_{-k}^{\dagger} \end{pmatrix}, \qquad (2)$$

where $\sigma_3 = \text{diag}(\mathbb{I}, -\mathbb{I})$ is a Pauli matrix acting on the particlehole degree of freedom.

Beyond linear spin-wave theory, the quasiparticle energies correspond to poles of the (retarded) single-magnon propagator

$$\boldsymbol{G}^{\mathrm{R}}(\boldsymbol{k},\omega) = \left[\omega + i0^{+} - \boldsymbol{\sigma}_{3}\left(\boldsymbol{S}\boldsymbol{M}_{\boldsymbol{k}} + \boldsymbol{\Sigma}^{\mathrm{R}}(\boldsymbol{k},\omega)\right)\right]^{-1}\boldsymbol{\sigma}_{3}, \quad (3)$$

where $\Sigma^{R}(\mathbf{k}, \omega)$ is the retarded (thermal) self-energy matrix. We note that the matrix in Eq. (3) has both sublattice indices and indices that track the normal and anomalous components [43, 57, 58]. We may then include leading order effects of spinwave interactions perturbatively by computing the self-energy to $O(S^{0})$ in the 1/S expansion [59]. This contribution to the self-energy can be represented as the sum of three one-particle irreducible diagrams [60]

$$\Sigma^{R} = - + - + - - + - - , (4)$$

constructed from both three- and four-magnon interaction vertices. In general, these diagrams carry a conserved wavevector and frequency (\mathbf{k}, ω) , and sublattice indices, and include both normal and anomalous propagators [56, 58]. Note that in order to incorporate thermal effects into the self-energy, we must take the limit $1/S \rightarrow 0$ while keeping T/S fixed [56].

Next, we consider a gapless PG mode, which falls into one of two categories depending on its spectral properties, referred to as "type I" and "type II" [43, 61]. Throughout this paper, we assume that this mode is located at the Brillouin zone center $(\mathbf{k} = \mathbf{0})$, although this assumption can be relaxed in general. At the level of linear spin-wave theory, a type I PG mode has linear dispersion, $\epsilon_{\mathbf{k}} \propto |\mathbf{k}|$, while the type II mode disperses quadratically, $\epsilon_{\mathbf{k}} \propto |\mathbf{k}|^2$, as depicted in Fig. 1. For a type I mode, there is a single linearly independent eigenvector V_0 corresponding to a zero eigenvalue. In this case, the PG gap appears at $O(S^{1/2})$ as

$$\Delta(T) = S^{1/2} \sqrt{V_0^{\dagger} \left[\Sigma_0^{\mathrm{R}}(T) \sigma_3 M_0 + M_0 \sigma_3 \Sigma_0^{\mathrm{R}}(T) \right] V_0}, \quad (5)$$

where $\Sigma_0^{\text{R}}(T) \equiv \Sigma^{\text{R}}(0, 0)$ is introduced to make the temperature dependence of the self-energy explicit. For a type II mode, there are two linearly independent eigenvectors, V_0 and W_0 , corresponding to a zero eigenvalue. In this case, the PG gap appears at $O(S^0)$ as

$$\Delta(T) = S^0 \sqrt{\left(V_0^{\dagger} \boldsymbol{\Sigma}_0^{\mathsf{R}}(T) V_0\right)^2 - \left|V_0^{\dagger} \boldsymbol{\Sigma}_0^{\mathsf{R}}(T) W_0\right|^2}.$$
 (6)

We refer the interested reader to the Supplementary Material of Ref. [43] for a detailed derivation of Eqs. (5,6). With the PG gap, the low energy dispersion now takes the form $\sqrt{v^2 |k|^2 + \Delta^2}$ for a type I mode and $v|k|^2 + \Delta$ for a type II mode, as depicted in Fig. 1. The gap then acquires a temperature dependence $\Delta \rightarrow \Delta(T)$ from the self-energy $\Sigma^{R}(k, \omega)$ in Eqs. (5,6).

Curvature formula—In previous work characterizing the PG gap in both the quantum zero-temperature [43] and classical finite-temperature [54] scenarios, it was argued that the gap at leading order can be calculated exactly by computing the curvature of the linear spin-wave free energy. Here, we develop a similar formalism analogous to Refs. [43, 54] that allows us to calculate the PG gap at $O(S^0)$ using only the linear spin-wave energy eigenvalues at O(S), circumventing the need to carry out tedious calculations of the magnon self-energy.

The derivation of such a formula resembles the zerotemperature proof in Ref. [43], which can be generalized to finite temperature. Consider a local reference frame $(\hat{\boldsymbol{e}}_{x,\alpha}, \hat{\boldsymbol{e}}_{y,\alpha}, \hat{\boldsymbol{e}}_{0,\alpha})$ where $\hat{\boldsymbol{e}}_{0,\alpha}$ is the classical ordering direction, $\hat{\boldsymbol{e}}_{y,\alpha}$ is the soft mode direction, and $\hat{\boldsymbol{e}}_{x,\alpha} = \hat{\boldsymbol{e}}_{y,\alpha} \times \hat{\boldsymbol{e}}_{0,\alpha}$. We may parametrize the soft mode by an angle ϕ (along with its canonically conjugate pair θ) and define the rotated Hamiltonian $\mathcal{H}(\phi, \theta) = U(\phi, \theta)^{\dagger} H U(\phi, \theta)$ where $U(\phi, \theta)$ is a unitary transformation that generates the soft modes of the spin configurations. Formally, this modifies the local frame as

$$\begin{pmatrix} \hat{\boldsymbol{\ell}}_{x,\alpha}(\phi,\theta) \\ \hat{\boldsymbol{\ell}}_{y,\alpha}(\phi,\theta) \\ \hat{\boldsymbol{\ell}}_{0,\alpha}(\phi,\theta) \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi\sin\theta & \sin\phi\cos\theta \\ 0 & \cos\theta & \sin\theta \\ -\sin\phi & -\cos\phi\sin\theta & \cos\phi\cos\theta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\ell}}_{x,\alpha} \\ \hat{\boldsymbol{\ell}}_{y,\alpha} \\ \hat{\boldsymbol{\ell}}_{0,\alpha} \end{pmatrix}.$$
(7)

We subsequently expand the rotated Hamiltonian to O(S) to obtain

$$\mathcal{H}(\phi,\theta) = S(S+1)N\epsilon_{\rm cl}(\theta) + SN\epsilon_{\rm qu}(\phi,\theta) + S\sum_{\boldsymbol{k},\alpha}\epsilon_{\boldsymbol{k},\alpha}(\phi,\theta)b^{\dagger}_{\boldsymbol{k},\alpha}b_{\boldsymbol{k},\alpha},$$
(8)

where $S^2 \epsilon_{cl}$ is the classical ground state energy (per spin) and $S(\epsilon_{cl} + \epsilon_{qu})$ is the quantum zero-point energy (per spin), and $S\epsilon_{k,\alpha}$ is the linear spin-wave dispersion [56]. Equation (8) effectively parametrizes the magnons with respect to the zero-mode subspace. For a type I mode, only ϕ corresponds to a soft mode at the classical level, so the classical configuration energy depends on θ . When $\theta \neq 0$, the spin configuration is classically unstable, implying that the zero-point and spin-wave energies are not well-defined. For a type II mode, both angles are classically soft, so $\epsilon_{cl}(\theta) = \epsilon_{cl}(0)$. In general, the free

energy per spin at O(S) is

$$f(\phi, \theta) = S(S+1)\epsilon_{\rm cl}(\theta) + S\epsilon_{\rm qu}(\phi, \theta) + \frac{k_{\rm B}T}{N} \sum_{k,\alpha} \ln\left(1 - e^{-S\epsilon_{k,\alpha}(\phi, \theta)/k_{\rm B}T}\right).$$
(9)

We may now establish an equivalence between the effective Hamiltonian to $O(S^1)$ in Eq. (8) and the PG gap using a sum rule for the magnon spectral function

$$\frac{1}{SN} \left\langle \left(\frac{\partial^2 \mathcal{H}}{\partial \lambda_{\mu} \partial \lambda_{\nu}} \right)_0 \right\rangle = U_{\mu}^{\dagger} \sigma_3 \left[\int_{-\infty}^{\infty} d\omega \, \omega \, \mathcal{A}(\mathbf{0}, \omega) \right] \sigma_3 U_{\nu},$$
(10)

where $(\cdots)_0$ is used to denote evaluation at $\phi = \theta = 0$, $\mu, \nu = \phi, \theta$ label the two angles, $\lambda_{\phi} = \phi$, $\lambda_{\theta} = \theta$, and $\mathcal{A}(\mathbf{0}, \omega) \equiv \frac{1}{2i} \left(\mathbf{G}^{\mathsf{R}}(\mathbf{0}, \omega) - \mathbf{G}^{\mathsf{R}}(\mathbf{0}, \omega)^{\dagger} \right)$ is the single-magnon spectral function evaluated at temperature *T*. The vectors U_{μ} are defined as $U_{\phi} \equiv i(W_0 - V_0)/\sqrt{2}$, $U_{\theta} \equiv (W_0 + V_0)/\sqrt{2}$, where V_0 and W_0 are vectors that span the zero-mode subspace in linear spin wave theory [43, 56]. Here, $\langle \cdots \rangle$ denotes the thermal expectation value at a fixed temperature *T*.

The right-hand side of Eq. (10) is related to the PG gap at $O(S^0)$, while the left-hand side is related to curvature of the linear-spin wave energies at O(S) and $O(S^2)$ [56], paving the way for bypassing the need of performing self-energy calculations. In particular, the thermal gap at leading order is given by the formula

$$\Delta(T) = \begin{cases} S^{1/2} \sqrt{\left(\frac{\partial^2 \epsilon_{\rm cl}}{\partial \theta^2}\right)_0 g_{\phi\phi}} & \text{(type I)} \\ S^0 \sqrt{g_{\phi\phi} g_{\theta\theta}} - g_{\phi\theta}^2 & \text{(type II)}, \end{cases}$$
(11)

where

 $g_{\mu\nu}(T) \equiv \frac{1}{S} \left[\left(\frac{\partial^2 f}{\partial \lambda_{\mu} \partial \lambda_{\nu}} \right)_0 + K_{\mu\nu} \right], \tag{12}$

and

$$K_{\mu\nu} \equiv \frac{S^2}{4k_{\rm B}TN} \sum_{\boldsymbol{k},\alpha} \left(\frac{\partial \epsilon_{\boldsymbol{k},\alpha}}{\partial \lambda_{\mu}}\right)_0 \left(\frac{\partial \epsilon_{\boldsymbol{k},\alpha}}{\partial \lambda_{\nu}}\right)_0 \operatorname{csch}^2 \left(\frac{S \epsilon_{\boldsymbol{k},\alpha}}{2k_{\rm B}T}\right).$$
⁽¹³⁾

As $T \to 0$, $\frac{S^2}{k_{\rm B}T} \operatorname{csch}^2\left(\frac{s\,\epsilon_{k,\alpha}}{2k_{\rm B}T}\right) \to \frac{4k_{\rm B}T}{\epsilon_{k,\alpha}^2}$ for small $\epsilon_{k,\alpha}$, while $\operatorname{csch}^2\left(\frac{S\,\epsilon_{k,\alpha}}{2k_{\rm B}T}\right) \to 0$ exponentially otherwise. Therefore $K_{\mu\nu}$

vanishes in the zero temperature limit. At high temperature, $K_{\mu\nu} \propto T$, so this term does not affect the scaling proposed in Ref. [62] for the classical PG gap. Note that since T/S is kept fixed in perturbation theory, $g_{\mu\nu}$ is formally of $O(S^0)$ whenever both μ and ν correspond to soft modes. The derivatives in Eq. (13) can be readily calculated using the Hellmann-Feynman theorem [63]. More details regarding the proof of Eq. (11) and the calculation of Eq. (13) are provided in the Supplemental Material [56]. The formula for the PG gap in Eq. (11) is a finite-temperature generalization of the curvature formula in Ref. [43].



FIG. 2. PG gap for the Heisenberg-compass model Eq. (14) on a simple cubic lattice with (a) $\xi = 0$ (type I mode) and (b) $\xi = 1$ (type II mode). In both cases, we use K/J = 0.5 and S = 1/2.

Models— As a first example, we consider a ferromagnetic Heisenberg-compass model on the simple cubic lattice, defined by the Hamiltonian

$$H = -\sum_{r} \left[J \sum_{\delta = \mathbf{x}, \mathbf{y}, \mathbf{z}} S_{r} \cdot S_{r+\delta} + K \left(S_{r}^{x} S_{r+x}^{x} + S_{r}^{y} S_{r+y}^{y} + \xi S_{r}^{z} S_{r+z}^{z} \right) \right], \qquad (14)$$

where S_r is a spin-*S* operator at site r and $\delta = x, y, z$ are the nearest-neighbor bonds, and J, K > 0. For K = 0, Eq. (14) is simply the Heisenberg ferromagnet, with collinear ground states related by a global SU(2) symmetry. When K > 0, there is no longer an exact continuous symmetry. In the following, we consider the cases of $\xi = 0$ and $\xi = 1$.

When $\xi = 0$, the classical ground state corresponds to a collinear ferromagnet with magnetization along any direction in the $\hat{x} - \hat{y}$ plane. In contrast to the (K = 0) Heisenberg model, this degeneracy is *accidental*. This accidental U(1) degeneracy is lifted by quantum and thermal fluctuations to select one of four magnetization directions along the $\pm \hat{x}, \pm \hat{y}$ directions. In this case, the associated PG mode is of type I.

When $\xi = 1$, the classical ground state is a collinear ferromagnet with arbitrary magnetization direction, signaling an accidental O(3) degeneracy. The degeneracy is lifted by fluctuations to select one of six magnetization directions along the cubic axes $\pm \hat{x}, \pm \hat{y}, \pm \hat{z}$. In this latter case, the associated PG mode is of type II [56].

The PG gap at $O(S^0)$ for Eq. (14) is depicted in Fig. 2 over a range of temperature in the case of both a type I ($\xi = 0$) and type II ($\xi = 1$) mode. We find an excellent quantitative agreement between the gap calculated using the curvature formula in Eq. (11) and the self-energy calculation of Eqs. (5,6). We emphasize that these two calculations are carried out independently of one another, and there are no free parameters introduced to make them agree. We note that the gap calculation is relatively simple for this model, as the lack of three-magnon interactions implies that $K_{\mu\nu} = 0$ in Eq. (12) [56]. In both cases, there is a zero-temperature contribution due to the gap Δ_0 induced by quantum ObD (i.e. zero-point fluctuations) [43]. At low-temperature, the leading thermal contribution to the gap is proportional to T^4 for a type I mode and $T^{5/2}$ for a type II mode.

Next, we discuss an application to the XY pyrochlore antiferromagnet $Er_2Ti_2O_7$, which is arguably one of the best material candidates for ObD [39, 49, 64]. The spin-orbit entangled electronic states of Er^{3+} are subject to a highly anisotropic crystal-field producing a ground doublet described using an effective (pseudo) spin- $\frac{1}{2}$ with the $Er^{3+}-E^{3+}$ interactions described by an anisotropic exchange-like model [39, 65, 66]

$$H = \sum_{\langle i,j \rangle} \left[J_{zz} S_i^z S_j^z - J_{\pm} \left(S_i^+ S_j^- + S_i^- S_j^+ \right) + J_{\pm\pm} \left(\gamma_{ij} S_i^+ S_j^+ + \text{H.c.} \right) + J_{z\pm} \left(\zeta_{ij} \left[S_i^z S_j^+ + S_i^+ S_j^z \right] + \text{H.c.} \right) \right].$$
(15)

Here $\langle i, j \rangle$ denotes the sum over nearest-neighbor bonds, and γ_{ij}, ζ_{ij} are bond-dependent phase factors uniquely determined by the lattice geometry and symmetries of the single-ion wavefunctions [66]. The four nearest-neighbour J_{uv} couplings in Eq. (15) have been fitted to reproduce inelastic neutron scattering data on Er₂Ti₂O₇ [39]. The classical ground states are non-collinear antiferromagnetic configurations of spins lying in the local XY planes perpendicular to the local $\langle 111 \rangle$ cubic axes of the pyrochlore lattice [66], parametrized by an accidental U(1) degeneracy [23, 67]. Below $T_c \approx 1.17$ K [35], ObD selects one of the six " ψ_2 " states, leading to a type I PG mode [39, 66].

The PG gap for $\text{Er}_2\text{Ti}_2\text{O}_7$ is depicted in Fig. 3 over a range of temperatures up to T = 700mK. We find a zero-temperature contribution to the gap of $\Delta_0 = 31.1 \ \mu\text{eV}$, consistent with the gap calculated in Ref. [43], and somewhat smaller than the experimentally determined values of 43 μeV [68] and 53 μeV [49]. At low-temperature, the gap scales proportional to T^4 , similar to the type I mode in Fig. 2(a).

Discussion—The curvature formula in Eq. (11) directly implies that the PG gap satisfies a universal scaling relation



FIG. 3. PG gap for the pyrochlore antiferromagnet $\text{Er}_2\text{Ti}_2\text{O}_7$, calculated using the exchange couplings from Ref. [39]. The classical ground states are parametrized by an accidental U(1) degeneracy, corresponding to a type I mode. The gap is plotted up to T = 700 mK.

as a function of temperature. By noting that the spin-wave dispersion is linear ($\epsilon_k \sim |\mathbf{k}|$) for a type I mode and quadratic ($\epsilon_k \sim |\mathbf{k}|^2$) for a type II mode, a straightforward dimensional analysis of Eq. (12) implies that the leading thermal correction generically satisfies

$$\Delta(T) - \Delta_0 \propto \begin{cases} S^{1/2} \left(\frac{T}{S}\right)^{d+1} & \text{(type I)} \\ S^0 \left(\frac{T}{S}\right)^{d/2+1} & \text{(type II)}, \end{cases}$$
(16)

where *d* is the spatial dimension, and is consistent with the results in Figs. 2 and 3. We propose that this universal temperature-dependence of the gap *increasing* with temperature at $0 < T \ll T_c$, where T_c is the critical temperature, can serve as experimental evidence for ObD in a spin systems that does not rely on fitting exchange couplings to a microscopic model.

The ability to calculate the PG gap using only linear spinwave theory [Eq. (11)] provides a substantial simplification when compared to the tedious calculation of of the magnon self-energy, particularly in cases with non-collinear order. In addition, the presence of a gapless mode in the linear spinwave spectrum renders a direct calculation of the self-energy infrared divergent when d < 3 [69], drawing an analogy to the Hohenberg–Mermin–Wagner theorem [70–73]. To circumvent this divergence, one could employ a more involved, self-consistent approach to calculate the self-energy, such as those described in Refs. [54, 74–77]. Alternatively, the curvature formula for the gap in Eq. (11) is well-defined for d < 3, avoiding the need to regularize an infrared divergence in the first place.

Our observation that the PG gap obeys a universal, temperature-dependent scaling relation can serve as a framework to diagnose ObD in real materials in a way that is independent of the microscopic spin Hamiltonian describing the material. Leading candidates where ObD may be present include the compounds $Er_2Ti_2O_7$ [49, 64, 68], CoTiO₃ [51], $Sr_2Cu_3O_4Cl_2$ [52], and $Fe_2Ca_3(GeO_4)_3$ [53]. It remains unclear, however, whether the small gap measured in these materials is truly a PG gap, as it is possible that such a gap can arise from further neighbor exchange, anisotropic spin interactions or higher-order spin interactions such as a biguadratic interaction. The experimental observation of a temperaturedependent gap such as in Eq. (16) would aid in resolving these questions. The experimental details of how to resolve and characterize small thermal corrections to the PG gap remain an open problem. For the pyrochlore material $Er_2Ti_2O_7$, the thermal correction to the gap up to T = 700 mK is predicted to be $\Delta(700 \text{ mK}) - \Delta_0 = 2.9 \ \mu\text{eV}$ (see Fig. 3). This will likely be challenging to measuring experimentally with high resolution in temperature as this energy scale is at the lower end of what is currently accessible using high-resolution inelastic neutron backscattering measurements [78]. In addition, there will be a crossover temperature, T^* , where thermal fluctuations and the progressive restoration of symmetry as T_c is approached from below, will take over and make $\Delta(T)$ begin decreasing. The treatment of corrections to $\Delta(T)$ arising from pre-critical fluctuations is beyond the scope of this work, as we focus only on $\Delta(T)$ in the limit $T \ll T_c$. Nonetheless, our work provides a robust theoretical framework to guide future experimental efforts in this exciting field of research.

Note—In the process of finalizing this manuscript, we became aware of a very recent preprint [79] that reports calculations of the PG gap at nonzero temperature similar to those presented here.

Acknowledgements—We thank Michael Burke, Anton Burkov and Griffin Howson for useful technical discussions, and Jason Gardner, Bruce Gaulin and Romain Sibille for discussions about inelastic neutron backscattering. A. H. acknowledges support from the NSERC of Canada CGS-D Scholarship. S. K. acknowledges financial support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy through the Würzburg-Dresden Cluster of Excellence on Complexity and Topology in Quantum Matter – ct.qmat (EXC 2147, project-id 390858490). The work at the U. of Waterloo and the U. of Windsor was supported by the NSERC of Canada (M. J. P. G. and J. G. R.) and the Canada Research Chair program (M. J. P. G., Tier 1). This research was enabled in part by computing resources provided by the Digital Research Alliance of Canada.

- [1] F. London, The λ -Phenomenon of Liquid Helium and the Bose-Einstein Degeneracy, Nature 141, 643 (1938).
- [2] L. Balents, Spin liquids in frustrated magnets, Nature 464, 199 (2010).
- [3] M. J. P. Gingras and P. A. McClarty, Quantum spin ice: A search for gapless quantum spin liquids in pyrochlore magnets, Rep. Prog. Phys. 77, 056501 (2014).
- [4] T. Imai and Y. S. Lee, Do quantum spin liquids exist?, Phys. Today 69, 30 (2016).

- [5] L. Savary and L. Balents, Quantum spin liquids: A review, Reports on Progress in Physics 80, 016502 (2017).
- [6] J. Knolle and R. Moessner, A field guide to spin liquids, Annu. Rev. Condens. Matter Phys. 10, 451 (2019).
- [7] J. Villain, Insulating spin-glasses, Zeitschrift f
 ür Physik B-Condensed Matter 33, 31 (1979).
- [8] R. Moessner and J. T. Chalker, Low-temperature properties of classical geometrically frustrated antiferromagnets, Phys. Rev. B 58, 12049 (1998).
- [9] D. Lozano-Gómez, V. Noculak, J. Oitmaa, R. R. P. Singh, Y. Iqbal, J. Reuther, and M. J. P. Gingras, Competing gauge fields and entropically driven spin liquid to spin liquid transition in non-Kramers pyrochlores, Proc. Natl. Acad. Sci. U.S.A. 121, e2403487121 (2024).
- [10] D. Lozano-Gómez, O. Benton, M. J. P. Gingras, and H. Yan, An atlas of classical pyrochlore spin liquids, arXiv:2411.03547 [cond-mat.str-el] 10.48550/arXiv.2411.03547 (2024).
- [11] Note that the existence of an accidental degeneracy by no means implies this intermediate scenario.
- [12] J. Villain, R. Bidaux, J.-P. Carton, and R. Conte, Order as an effect of disorder, J. Phys. 41, 1263 (1980).
- [13] E. F. Shender, Antiferromagnetic garnets with fluctuationally interacting sublattices, Sov. Phys. JETP 56, 178 (1982).
- [14] C. L. Henley, Ordering due to disorder in a frustrated vector antiferromagnet, Phys. Rev. Lett. 62, 2056 (1989).
- [15] C. L. Henley, Ordering by disorder: Ground-state selection in fcc vector antiferromagnets, J. Appl. Phys. 61, 3962 (1987).
- [16] S. Prakash and C. L. Henley, Ordering due to disorder in dipolar magnets on two-dimensional lattices, Phys. Rev. B 42, 6574 (1990).
- [17] J. N. Reimers and A. J. Berlinsky, Order by disorder in the classical Heisenberg kagomé Antiferromagnet, Phys. Rev. B 48, 9539 (1993).
- [18] V. Noculak, D. Lozano-Gómez, J. Oitmaa, R. R. P. Singh, Y. Iqbal, M. J. P. Gingras, and J. Reuther, Classical and quantum phases of the pyrochlore $S = \frac{1}{2}$ magnet with Heisenberg and Dzyaloshinskii-Moriya interactions, Phys. Rev. B **107**, 214414 (2023).
- [19] A. Hickey, D. Lozano-Gómez, and M. J. P. Gingras, Orderby-disorder without quantum zero-point fluctuations in the pyrochlore Heisenberg ferromagnet with Dzyaloshinskii-Moriya interactions, Phys. Rev. B 111, 184434 (2025).
- [20] D. Bergman, J. Alicea, E. Gull, S. Trebst, and L. Balents, Orderby-disorder and spiral spin-liquid in frustrated diamond-lattice antiferromagnets, Nat. Phys. 3, 487 (2007).
- [21] G.-W. Chern, Pyrochlore antiferromagnet with antisymmetric exchange interactions: critical behavior and order from disorder, arXiv:1008.3038 [cond-mat.str-el] (2010).
- [22] M. V. Gvozdikova, P.-E. Melchy, and M. E. Zhitomirsky, Magnetic phase diagrams of classical triangular and kagome antiferromagnets, J. Phys. Condens. Matter 23, 164209 (2011).
- [23] P. A. McClarty, P. Stasiak, and M. J. P. Gingras, Order-by-disorder in the XY pyrochlore antiferromagnet, Phys. Rev. B 89, 024425 (2014).
- [24] N. Francini, L. Janssen, and D. Lozano-Gómez, Higher-rank spin liquids and spin nematics from competing orders in pyrochlore magnets, Phys. Rev. B 111, 085140 (2025).
- [25] J. R. Tessman, Magnetic Anisotropy at 0°K, Phys. Rev. 96, 1192 (1954).
- [26] E. Belorizky, R. Casalegno, and J. J. Niez, Calculation of the Spin Wave Energy Gap at *k* = 0 for a Simple Cubic Ferromagnet with Anisotropic Exchange Interactions, Phys. Status Solidi B 102, 365 (1980).
- [27] P. Chandra and B. Doucot, Possible spin-liquid state at large

S for the frustrated square Heisenberg lattice, Phys. Rev. B **38**, 9335 (1988).

- [28] K. Kubo and T. Kishi, Ordering Due to Quantum Fluctuations in the Frustrated Heisenberg Model, J. Phys. Soc. Jpn. 60, 567 (1991).
- [29] S. Sachdev, Kagomé and triangular-lattice Heisenberg antiferromagnets: Ordering from quantum fluctuations and quantumdisordered ground states with unconfined bosonic spinons, Phys. Rev. B 45, 12377 (1992).
- [30] A. V. Chubukov and D. I. Golosov, Quantum theory of an antiferromagnet on a triangular lattice in a magnetic field, J. Phys. Condens. Matter 3, 69 (1991).
- [31] A. Chubukov, Order from Disorder in a Kagomé Antiferromagnet, Phys. Rev. Lett. 69, 832 (1992).
- [32] C. L. Henley, Selection by Quantum Fluctuations of Dipolar Order in a Diamond Lattice, Phys. Rev. Lett. 73, 2788 (1994).
- [33] P. Lecheminant, B. Bernu, C. Lhuillier, and L. Pierre, $J_1 J_2$ quantum Heisenberg antiferromagnet on the triangular lattice: A group-symmetry analysis of order by disorder, Phys. Rev. B **52**, 6647 (1995).
- [34] R. R. Sobral and C. Lacroix, Order by disorder in the pyrochlore antiferromagnets, Solid State Communications 103, 407 (1997).
- [35] J. D. M. Champion, M. J. Harris, P. C. W. Holdsworth, A. S. Wills, G. Balakrishnan, S. T. Bramwell, E. Čižmár, T. Fennell, J. S. Gardner, J. Lago, D. F. McMorrow, M. Orendáč, A. Orendáčová, D. M. Paul, R. I. Smith, M. T. F. Telling, and A. Wildes, Er₂Ti₂O₇: Evidence of quantum order by disorder in a frustrated antiferromagnet, Phys. Rev. B 68, 020401 (2003).
- [36] G. Baskaran, D. Sen, and R. Shankar, Spin-S Kitaev model: Classical ground states, order from disorder, and exact correlation functions, Phys. Rev. B 78, 115116 (2008).
- [37] J.-S. Bernier, M. J. Lawler, and Y. B. Kim, Quantum Order by Disorder in Frustrated Diamond Lattice Antiferromagnets, Phys. Rev. Lett. 101, 047201 (2008).
- [38] A. Mulder, R. Ganesh, L. Capriotti, and A. Paramekanti, Spiral order by disorder and lattice nematic order in a frustrated Heisenberg antiferromagnet on the honeycomb lattice, Phys. Rev. B 81, 214419 (2010).
- [39] L. Savary, K. A. Ross, B. D. Gaulin, J. P. C. Ruff, and L. Balents, Order by Quantum Disorder in Er₂Ti₂O₇, Phys. Rev. Lett. 109, 167201 (2012).
- [40] M. E. Zhitomirsky, M. V. Gvozdikova, P. C. W. Holdsworth, and R. Moessner, Quantum Order by Disorder and Accidental Soft Mode in Er₂Ti₂O₇, Phys. Rev. Lett. **109**, 077204 (2012).
- [41] A. L. Chernyshev and M. E. Zhitomirsky, Quantum Selection of Order in an XXZ Antiferromagnet on a Kagome Lattice, Phys. Rev. Lett. 113, 237202 (2014).
- [42] I. Rousochatzakis, J. Reuther, R. Thomale, S. Rachel, and N. B. Perkins, Phase Diagram and Quantum Order by Disorder in the Kitaev $K_1 K_2$ Honeycomb Magnet, Phys. Rev. X 5, 041035 (2015).
- [43] J. G. Rau, P. A. McClarty, and R. Moessner, Pseudo-Goldstone Gaps and Order-by-Quantum Disorder in Frustrated Magnets, Phys. Rev. Lett. 121, 237201 (2018).
- [44] B. Placke, R. Moessner, and O. Benton, Hierarchy of energy scales and field-tunable order by disorder in dipolar-octupolar pyrochlores, Phys. Rev. B 102, 245102 (2020).
- [45] G. Chen and X. Wang, Electron quasi-itinerancy intertwined with quantum order by disorder in pyrochlore iridate magnetism, Phys. Rev. Res. 2, 043273 (2020).
- [46] C. Liu, C.-J. Huang, and G. Chen, Intrinsic quantum Ising model on a triangular lattice magnet TmMgGao₄, Phys. Rev. Res. 2, 043013 (2020).
- [47] S. Khatua, S. Srinivasan, and R. Ganesh, State selection in

frustrated magnets, Phys. Rev. B 103, 174412 (2021).

- [48] S. Khatua, G. C. Howson, M. J. P. Gingras, and J. G. Rau, Ground state properties of the Heisenberg-compass model on the square lattice, Phys. Rev. B 110, 104426 (2024).
- [49] K. A. Ross, Y. Qiu, J. R. D. Copley, H. A. Dabkowska, and B. D. Gaulin, Order by Disorder Spin Wave Gap in the XY Pyrochlore Magnet Er₂Ti₂O₇, Phys. Rev. Lett. **112**, 057201 (2014).
- [50] C. L. Sarkis, J. G. Rau, L. D. Sanjeewa, M. Powell, J. Kolis, J. Marbey, S. Hill, J. A. Rodriguez-Rivera, H. S. Nair, D. R. Yahne, S. Säubert, M. J. P. Gingras, and K. A. Ross, Unravelling competing microscopic interactions at a phase boundary: A single-crystal study of the metastable antiferromagnetic pyrochlore Yb₂Ge₂O₇, Phys. Rev. B **102**, 134418 (2020).
- [51] M. Elliot, P. A. McClarty, D. Prabhakaran, R. D. Johnson, H. C. Walker, P. Manuel, and R. Coldea, Order-by-Disorder from Bond-Dependent Exchange and Intensity Signature of Nodal Quasiparticles in a Honeycomb Cobaltate, Nature Communications 12, 3936 (2021).
- [52] Y. J. Kim, A. Aharony, R. J. Birgeneau, F. C. Chou, O. Entin-Wohlman, R. W. Erwin, M. Greven, A. B. Harris, M. A. Kastner, I. Y. Korenblit, Y. S. Lee, and G. Shirane, Ordering due to Quantum Fluctuations in Sr₂Cu₃O₄Cl₂, Phys. Rev. Lett. 83, 852 (1999).
- [53] T. Brueckel, B. Dorner, A. G. Gukasov, V. P. Plakhty, W. Prandl, E. F. Shender, and O. P. Smirnow, Dynamical interaction of antiferromagnetic subsystems: a neutron scattering study of the spinwave spectrum of the garnet Fe₂Ca₃(GeO₄)₃, Z. Phys. B 72, 477 (1988).
- [54] S. Khatua, M. J. P. Gingras, and J. G. Rau, Pseudo-goldstone modes and dynamical gap generation from order by thermal disorder, Phys. Rev. Lett. 130, 266702 (2023).
- [55] T. Holstein and H. Primakoff, Field Dependence of the Intrinsic Domain Magnetization of a Ferromagnet, Phys. Rev. 58, 1098 (1940).
- [56] Refer to the Supplemental Material for details of spin-wave theory and the proof of the curvature formula.
- [57] J.-P. Blaizot and G. Ripka, *Quantum Theory of Finite Systems* (MIT Press, 1986).
- [58] Here, "normal" refers to the number-conserving blocks of Eq. (1) (i.e. terms proportional to $a^{\dagger}a$ and aa^{\dagger}) and "anomalous" refers to the number-non-conserving blocks (i.e. terms proportional to $a^{\dagger}a^{\dagger}$ and aa).
- [59] E. Rastelli, Statistical Mechanics of Magnetic Excitations: from Spin Waves to Stripes to Checkerboards (World Scientific, London, 2013).
- [60] J. W. Negele and H. Orland, *Quantum Many-particle Systems* (Westview Press, 1988).
- [61] H. Watanabe and H. Murayama, Unified Description of Nambu-Goldstone Bosons without Lorentz Invariance, Phys. Rev. Lett. 108, 251602 (2012).
- [62] S. Khatua, M. J. P. Gingras, and J. G. Rau, Pseudo-Goldstone Modes and Dynamical Gap Generation from Order by Thermal Disorder, Phys. Rev. Lett. 130, 266702 (2023).
- [63] R. P. Feynman, Forces in Molecules, Phys. Rev. 56, 340 (1939).
- [64] S. Petit, J. Robert, S. Guitteny, P. Bonville, C. Decorse, J. Ollivier, H. Mutka, M. J. P. Gingras, and I. Mirebeau, Order by disorder or energetic selection of the ground state in the XY pyrochlore antiferromagnet Er₂Ti₂O₇: An inelastic neutron scattering study, Phys. Rev. B **90**, 060410 (2014).
- [65] K. A. Ross, L. Savary, B. D. Gaulin, and L. Balents, Quantum excitations in quantum spin ice, Phys. Rev. X 1, 021002 (2011).
- [66] J. G. Rau and M. J. P. Gingras, Frustrated Quantum Rare-Earth Pyrochlores, Annual Review of Condensed Matter Physics 10, 357 (2019).

- [67] H. Yan, O. Benton, L. Jaubert, and N. Shannon, Theory of multiple-phase competition in pyrochlore magnets with anisotropic exchange with application to Yb₂Ti₂O₇, Er₂Ti₂O₇, and Er₂Sn₂O₇, Phys. Rev. B **95**, 094422 (2017).
- [68] E. Lhotel, J. Robert, E. Ressouche, F. Damay, I. Mirebeau, J. Ollivier, H. Mutka, P. Dalmas de Réotier, A. Yaouanc, C. Marin, C. Decorse, and S. Petit, Field-induced phase diagram of the XY pyrochlore antiferromagnet Er₂Ti₂O₇, Phys. Rev. B **95**, 134426 (2017).
- [69] S. Coleman, There are no Goldstone bosons in two dimensions, Commun. Math. Phys 31, 259 (1973).
- [70] N. D. Mermin and H. Wagner, Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models, Phys. Rev. Lett. 17, 1133 (1966).
- [71] P. C. Hohenberg, Existence of Long-Range Order in One and Two Dimensions, Phys. Rev. 158, 383 (1967).
- [72] V. L. Berezinskii, Destruction of Long-range Order in Onedimensional and Two-dimensional Systems having a Continuous Symmetry Group I. Classical Systems, Zh. Eksp. Teor. Fiz. 59, 907 (1970), [Sov. Phys. JETP 32, 493 (1971)].
- [73] V. L. Berezinskii, Destruction of Long-range Order in Onedimensional and Two-dimensional Systems Possessing a Continuous Symmetry Group II. Quantum Systems, Zh. Eksp. Teor. Fiz. **61**, 1144 (1971), [Sov. Phys. JETP **34**, 610 (1972)].
- [74] P. D. Loly, The Heisenberg ferromagnet in the selfconsistently renormalized spin wave approximation, J. Phys. C: Solid State Phys. 4, 1365 (1971).
- [75] W. T. Fuhrman, M. Mourigal, M. E. Zhitomirsky, and A. L. Chernyshev, Dynamical structure factor of quasi-twodimensional antiferromagnet in high fields, Phys. Rev. B 85, 184405 (2012).
- [76] R. Schick, O. Götze, T. Ziman, R. Zinke, J. Richter, and M. E. Zhitomirsky, Ground-state selection by magnon interactions in a fcc antiferromagnet, Phys. Rev. B 106, 094431 (2022).
- [77] C. A. Gallegos and A. L. Chernyshev, Magnon interactions in the quantum paramagnetic phase of CoNb₂O₆, Phys. Rev. B 109, 014424 (2024).
- [78] J. S. Gardner, G. Ehlers, A. Faraone, and V. G. Sakai, Highresolution neutron spectroscopy using backscattering and neutron spin-echo spectrometers in soft and hard condensed matter, Nature Reviews Physics 2, 103 (2020).
- [79] X. Lin and T. Shi, Pseudo-Goldstone Modes at Finite Temperature, arXiv:2505.07229v1 [cond-mat.str-el] (2025).

Supplemental Material for "Universal temperature-dependent power law excitation gaps in frustrated quantum spin systems harboring order-by-disorder"

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I. DETAILS OF SPIN WAVE THEORY

A. Holstein-Primakoff expansion

We consider a general bilinear spin model

$$H = \frac{1}{2} \sum_{\boldsymbol{rr'}} \sum_{\alpha \alpha'} \boldsymbol{S}_{\boldsymbol{r},\alpha}^{\mathsf{T}} \boldsymbol{J}_{\boldsymbol{r}-\boldsymbol{r'},\alpha \alpha'} \boldsymbol{S}_{\boldsymbol{r'},\alpha'},$$
(S1)

where *r* represents the position of a unit cell and α is a sublattice index. The associated spin wave theory can be obtained by expressing the spin-operators as Holstein-Primakoff bosons,

$$\boldsymbol{S}_{\boldsymbol{r},\alpha} = \left(\boldsymbol{S} - \boldsymbol{n}_{\boldsymbol{r},\alpha}\right) \boldsymbol{\hat{e}}_{\alpha,0} + \sqrt{S} \left[\left(1 - \frac{\boldsymbol{n}_{\boldsymbol{r},\alpha}}{2S}\right)^{1/2} \boldsymbol{a}_{\boldsymbol{r},\alpha} \boldsymbol{\hat{e}}_{\alpha,-} + \boldsymbol{a}_{\boldsymbol{r}\alpha}^{\dagger} \left(1 - \frac{\boldsymbol{n}_{\boldsymbol{r},\alpha}}{2S}\right)^{1/2} \boldsymbol{\hat{e}}_{\alpha,+} \right], \quad (S2)$$

where $n_{r,\alpha} = a_{r,\alpha}^{\dagger} a_{r,\alpha}$ and the vectors $\hat{\boldsymbol{e}}_{\alpha,\pm} \equiv (\hat{\boldsymbol{e}}_{\alpha,x} \pm i\hat{\boldsymbol{e}}_{\alpha,y})/\sqrt{2}$, $\hat{\boldsymbol{e}}_{\alpha,0}$ define a local reference frame with respect to the classical ordering direction $\hat{\boldsymbol{e}}_{\alpha,0}$ (i.e. $\hat{\boldsymbol{e}}_{\alpha,x} \times \hat{\boldsymbol{e}}_{\alpha,y} = \hat{\boldsymbol{e}}_{\alpha,0}$). We define the (Fourier transform of the) exchange interactions in this local frame as

$$\mathcal{J}_{\boldsymbol{k},\alpha\beta}^{\mu\nu} \equiv \sum_{\delta} e^{i\boldsymbol{k}\cdot\delta} \hat{\boldsymbol{e}}_{\alpha,\mu}^{\mathsf{T}} \boldsymbol{J}_{\delta,\alpha\beta} \hat{\boldsymbol{e}}_{\beta,\nu},\tag{S3}$$

where the sum is taken over all bonds δ between a site on sublattice α to a site on sublattice β . Expanding Eq. (S2) in powers of 1/S about an ordered state (typically a classical ground state), leads to the series representation $H = S^2 \sum_{n=0}^{\infty} S^{-n/2} H_n$. The magnon interactions to $O(S^0)$ are given by

$$H_0 = N\left(1 + \frac{1}{S}\right)\epsilon_{\rm cl},\tag{S4}$$

$$H_1 = \sqrt{N_c} \sum_{\alpha} \left[L^{\alpha} a^{\dagger}_{\mathbf{0},\alpha} + \overline{L}^{\alpha} a_{\mathbf{0},\alpha} \right],$$
(S5)

$$H_{2} = \frac{1}{2} \sum_{k} \sum_{\alpha\beta} \left[A_{k}^{\alpha\beta} a_{k,\alpha}^{\dagger} a_{k,\beta} + A_{-k}^{\beta\alpha} a_{-k,\alpha} a_{-k,\beta}^{\dagger} + B_{k}^{\alpha\beta} a_{k,\alpha}^{\dagger} a_{-k,\beta}^{\dagger} + \overline{B}_{k}^{\alpha\beta} a_{-k,\beta} a_{k,\alpha} \right],$$
(S6)

$$H_{3} = \frac{1}{2!\sqrt{N_{c}}} \sum_{\boldsymbol{k}\boldsymbol{q}} \sum_{\alpha\beta\mu} \left[T_{\boldsymbol{k},\boldsymbol{q}}^{\alpha\beta\mu} a_{\boldsymbol{k},\alpha}^{\dagger} a_{\boldsymbol{q},\beta}^{\dagger} a_{\boldsymbol{k}+\boldsymbol{q},\mu} + \overline{T}_{\boldsymbol{k},\boldsymbol{q}}^{\alpha\beta\mu} a_{\boldsymbol{k}+\boldsymbol{q},\mu}^{\dagger} a_{\boldsymbol{q},\beta} a_{\boldsymbol{k},\alpha} \right],$$
(S7)

$$H_{4} = \frac{1}{N_{c}} \sum_{\boldsymbol{k} \boldsymbol{q} \boldsymbol{Q}} \sum_{\alpha \beta \mu \nu} \left[\frac{1}{(2!)^{2}} V_{\boldsymbol{k}, \boldsymbol{q}, \boldsymbol{Q}}^{\alpha \beta \mu \nu} a_{\boldsymbol{k} + \boldsymbol{Q}, \alpha}^{\dagger} a_{\boldsymbol{q} - \boldsymbol{Q}, \beta}^{\dagger} a_{\boldsymbol{q}, \mu} a_{\boldsymbol{k}, \nu} + \frac{1}{3!} \left(D_{\boldsymbol{k}, \boldsymbol{q}, \boldsymbol{Q}}^{\alpha \beta \mu \nu} a_{\boldsymbol{k}, \alpha}^{\dagger} a_{\boldsymbol{q}, \beta}^{\dagger} a_{\boldsymbol{Q}, \mu}^{\dagger} a_{\boldsymbol{k} + \boldsymbol{q} + \boldsymbol{Q}, \nu} + \text{H.c.} \right) \right],$$
(S8)

where $N = N_c N_s$ is the number of spins in a system with N_c primitive unit cells, $NS^2 \epsilon_{cl}$ is the energy of the classical configuration, and the symmetrized interaction vertices are

$$\epsilon_{\rm cl} = \frac{1}{2N_s} \sum_{\alpha\beta} \mathcal{J}_{\mathbf{0},\alpha\beta}^{00},\tag{S9}$$

$$L^{\alpha} = \sum_{\beta} \mathcal{J}^{+0}_{\mathbf{0},\alpha\beta},\tag{S10}$$

$$A_{k}^{\alpha\beta} = \mathcal{J}_{k,\alpha\beta}^{+-} - \delta_{\alpha\beta} \sum_{\mu} \mathcal{J}_{\mathbf{0},\alpha\mu}^{00},$$
(S11)

$$B_k^{\alpha\beta} = \mathcal{J}_{k,\alpha\beta}^{++},\tag{S12}$$

$$T_{kq}^{\alpha\beta\mu} = -\delta_{\beta\mu}\mathcal{J}_{k,\alpha\beta}^{+0} - \delta_{\alpha\mu}\mathcal{J}_{q,\beta\alpha}^{+0},$$
(S13)

$$V_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{Q}}^{\alpha\beta\mu\nu} = \frac{1}{2} \delta_{\alpha\mu} \delta_{\beta\nu} \left(\mathcal{J}_{\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{Q},\alpha\beta}^{00} + \mathcal{J}_{-\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{Q},\beta\alpha}^{00} \right) + \frac{1}{2} \delta_{\alpha\nu} \delta_{\beta\mu} \left(+ \mathcal{J}_{\boldsymbol{Q},\alpha\beta}^{00} + \mathcal{J}_{-\boldsymbol{Q},\beta\alpha}^{00} \right)$$
$$= \frac{1}{2} \left(\delta_{\alpha} \delta_{\alpha} \mathcal{J}^{+-}_{-} + \delta_{\alpha} \delta_{\alpha} \mathcal{J}^{+-}_{-} + \delta_{\alpha} \delta_{\alpha} \mathcal{J}^{+-}_{-} \right)$$
(S14)

$$-\frac{1}{2} \left(\delta_{\beta\mu} \delta_{\beta\nu} \mathcal{J}_{k+Q,\alpha\beta} + \delta_{\alpha\mu} \delta_{\alpha\nu} \mathcal{J}_{q-Q,\beta\alpha} + \delta_{\alpha\beta} \delta_{\alpha\nu} \mathcal{J}_{q,\alpha\mu} + \delta_{\alpha\beta} \delta_{\alpha\mu} \mathcal{J}_{k,\alpha\nu} \right), \tag{S14}$$

$$D_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{Q}}^{\alpha\beta\mu\nu} = -\frac{1}{2} \left(\delta_{\mu\beta} \delta_{\nu\beta} \mathcal{J}_{\boldsymbol{k},\alpha\beta}^{++} + \delta_{\alpha\mu} \delta_{\alpha\nu} \mathcal{J}_{\boldsymbol{q},\beta\alpha}^{++} + \delta_{\alpha\beta} \delta_{\alpha\nu} \mathcal{J}_{\boldsymbol{Q},\mu\alpha}^{++} \right).$$
(S15)

The one-magnon vertex L^{α} vanishes so long as the classical configuration energy is at a local minimum.

B. Linear spin wave theory

The linear spin wave Hamiltonian results from truncating the Holstein-Primakoff expansion at O(S) about a configuration corresponding to a classical ground state

$$H = NS(S+1)\epsilon_{\rm cl} + S\sum_{k}\sum_{\alpha\alpha'} \left[A_{k}^{\alpha\alpha'} a_{k,\alpha}^{\dagger} a_{k,\alpha'} + \frac{1}{2} \left(B_{k}^{\alpha\alpha'} a_{k,\alpha}^{\dagger} a_{-k,\alpha'}^{\dagger} + \text{H.c.} \right) \right] + O(S^{1/2}).$$
(S16)

The linear spin wave energies and corresponding wavefunctions are determined by diagonalizing the $2N_s \times 2N_s$ boson Bogoliubov-de Gennes matrix [1]

$$\sigma_3 M_k \equiv \begin{pmatrix} A_k & B_k \\ -B_k^{\dagger} & -A_{-k}^{\dagger} \end{pmatrix},$$
(S17)

where $\sigma_3 = \text{diag}(\mathbb{I}, -\mathbb{I})$ is a block Pauli matrix. We may diagonalize Eq. (S17) by defining a new set of bosons $b_{k,\alpha}$, such that

$$\begin{pmatrix} a_k \\ a_{-k}^{\dagger} \end{pmatrix} = \mathcal{T}_k \begin{pmatrix} b_k \\ b_{-k}^{\dagger} \end{pmatrix}.$$
 (S18)

By requiring $\begin{bmatrix} b_{k,\alpha}, b_{k,\beta} \end{bmatrix} = 0$ and $\begin{bmatrix} b_{k,\alpha}, b_{k,\beta}^{\dagger} \end{bmatrix} = \delta_{\alpha\beta}$, we obtain the para-unitary condition [2]

$$\mathcal{T}_{k}^{-1} = \sigma_{3} \mathcal{T}_{k}^{\dagger} \sigma_{3}. \tag{S19}$$

The eigenvectors of Eq. (S17) come in pairs $V_{k,\alpha}$ and $W_{k,\alpha} = \sigma_1 \overline{V}_{-k,\alpha}$ [3], with eigenvalues $\pm \epsilon_{\pm k,\alpha}$, and make up the columns of the transformation matrix, i.e.

$$\mathcal{T}_{k} = \begin{pmatrix} | & | & | & | \\ V_{k,1} \cdots V_{k,N_{c}} & W_{k,1} \cdots W_{k,N_{c}} \\ | & | & | & | \end{pmatrix}.$$
 (S20)

Using Eq. (S19), the eigenvectors can be normalized to satisfy the para-orthogonality conditions

$$V_{k,\alpha}^{\dagger}\sigma_{3}V_{k,\beta} = +\delta_{\alpha\beta}, \qquad (S21)$$

$$W_{k,\alpha}^{\dagger}\sigma_{3}W_{k,\beta} = -\delta_{\alpha\beta}, \qquad (S22)$$

$$\boldsymbol{V}_{\boldsymbol{k},\alpha}^{\dagger}\boldsymbol{\sigma}_{\boldsymbol{3}}\boldsymbol{W}_{\boldsymbol{k},\beta} = 0. \tag{S23}$$

The linear spin wave Hamiltonian then takes the diagonal form

$$H = NS(S+1)\epsilon_{\rm cl} + NS\epsilon_{\rm qu} + \frac{S}{2}\sum_{k}\sum_{\alpha}\epsilon_{k,\alpha}b^{\dagger}_{k,\alpha}b_{k,\alpha} + O(S^{1/2}), \tag{S24}$$

where the quantum zero-point energy (per spin) is

$$S\epsilon_{\rm qu} \equiv \frac{S}{2N} \sum_{k} \sum_{\alpha} \epsilon_{k,\alpha}.$$
 (S25)

We may subsequently calculate the free energy (per spin) to O(S) at temperature T as

$$f(T) = S(S+1)\epsilon_{\rm cl} + S\epsilon_{\rm qu} + \frac{k_{\rm B}T}{N} \sum_{k} \sum_{\alpha} \ln\left(1 - e^{-S\epsilon_{k,\alpha}/k_{\rm B}T}\right) + O(S^{1/2}).$$
(S26)

Next, we discuss the circumstance of interest, where the linear spin wave spectrum contains zero modes. This implies that the matrix $\sigma_3 M_k$ is positive *semi-definite*. We assume, without loss of generality, that there is a single zero mode at the Brillouin zone center (k = 0). This mode can be classified based on the spectral properties of $M_0 \equiv M_0$. In particular, M_0 has either one or two linearly independent eigenvectors corresponding to the zero mode, which we refer to as *type I* and *type II* respectively. We define V_0 and W_0 to be the vectors that span the zero mode subspace, while simultaneously satisfying the normalization conditions Eqs. (S21-S23). The projection of M_0 into

the zero mode subspace is then given by [4]

$$\begin{pmatrix} \mathbf{V}_{0}^{\dagger} \mathbf{M}_{0} \mathbf{V}_{0} & \mathbf{V}_{0}^{\dagger} \mathbf{M}_{0} \mathbf{W}_{0} \\ \mathbf{W}_{0}^{\dagger} \mathbf{M}_{0} \mathbf{V}_{0} & \mathbf{W}_{0}^{\dagger} \mathbf{M}_{0} \mathbf{W}_{0} \end{pmatrix} = \begin{cases} \mathbf{V}_{0}^{\dagger} \mathbf{M}_{0} \mathbf{V}_{0} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{(type I)} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & & \text{(type II)} \end{cases}$$
(S27)

A more detailed treatment of non-interacting bosons with zero modes can be found in Refs. [1, 4].

C. Non-linear spin wave theory at finite temperature

To incorporate effects of spin wave interactions at leading order in 1/S and at finite temperature, we use the imaginary time formalism. To this effect, we assume the O(S) Hamiltonian in Eq. (S16) is solvable, and treat the three- and four-body interactions perturbatively to $O(S^0)$ [5]. We are interested in the single-magnon spectrum, encoded in the thermal Green's functions

$$\mathcal{G}_{\alpha\beta}^{-+}(\boldsymbol{k},i\omega_n) = \int_0^{(k_{\rm B}T)^{-1}} \mathrm{d}\tau \, e^{i\omega_n\tau} \left\langle \mathbb{T}a_{\boldsymbol{k},\alpha}(\tau)a_{\boldsymbol{k},\beta}^{\dagger}(0) \right\rangle,\tag{S28}$$

$$\mathcal{G}_{\alpha\beta}^{+-}(\boldsymbol{k},i\omega_n) = \int_0^{(k_{\rm B}T)^{-1}} \mathrm{d}\tau \, e^{i\omega_n\tau} \left\langle \mathbb{T}a_{-\boldsymbol{k},\alpha}^{\dagger}(\tau)a_{-\boldsymbol{k},\beta}(0) \right\rangle,\tag{S29}$$

$$\mathcal{G}_{\alpha\beta}^{++}(\boldsymbol{k},i\omega_n) = \int_0^{(k_{\rm B}T)^{-1}} \mathrm{d}\tau \, e^{i\omega_n\tau} \left\langle \mathbb{T}a_{-\boldsymbol{k},\alpha}^{\dagger}(\tau)a_{\boldsymbol{k},\beta}^{\dagger}(0) \right\rangle,\tag{S30}$$

$$\mathcal{G}_{\alpha\beta}^{--}(\boldsymbol{k},i\omega_n) = \int_0^{(k_{\rm B}T)^{-1}} \mathrm{d}\tau \, e^{i\omega_n\tau} \left\langle \mathbb{T}a_{\boldsymbol{k},\alpha}(\tau)a_{-\boldsymbol{k},\beta}(0) \right\rangle,\tag{S31}$$

where α, β label the sublattice structure, $\omega_n = \frac{2\pi n}{k_B T}$ is a bosonic Matsubara frequency, \mathbb{T} is the (imaginary) time-ordering operator [6], $\langle \cdots \rangle$ denotes the thermal average with respect to the equilibrium density matrix $\rho = Z^{-1} e^{-H/k_B T}$ [7]. This can be organized more compactly into a Bogoliubov-de Gennes block matrix, analogous to Eq. (S17)

$$\boldsymbol{\mathcal{G}}(\boldsymbol{k},i\omega_n) \equiv \begin{pmatrix} \boldsymbol{\mathcal{G}}^{-+}(\boldsymbol{k},i\omega_n) & \boldsymbol{\mathcal{G}}^{--}(\boldsymbol{k},i\omega_n) \\ \boldsymbol{\mathcal{G}}^{++}(\boldsymbol{k},i\omega_n) & \boldsymbol{\mathcal{G}}^{+-}(\boldsymbol{k},i\omega_n) \end{pmatrix} = \left[-i\omega_n + \boldsymbol{\sigma}_3 \left(S\boldsymbol{M}_{\boldsymbol{k}} + \boldsymbol{\Sigma}(\boldsymbol{k},i\omega_n) \right) \right]^{-1} \boldsymbol{\sigma}_3, \quad (S32)$$

where $\Sigma(\mathbf{k}, i\omega_n)$ is the imaginary time self-energy that encodes interactions. The single-magnon dispersion then corresponds to poles of the (real time) retarded Green's function, related to the thermal Green's function via analytic continuation

$$\boldsymbol{G}^{\mathrm{R}}(\boldsymbol{k},\omega) = -\boldsymbol{\mathcal{G}}(\boldsymbol{k},i\omega_n \to \omega + i0^+) = \left[\omega + i0^+ - \boldsymbol{\sigma}_3\left(S\boldsymbol{M}_{\boldsymbol{k}} + \boldsymbol{\Sigma}^{\mathrm{R}}(\boldsymbol{k},\omega)\right)\right]^{-1}\boldsymbol{\sigma}_3, \qquad (S33)$$

with the retarded self-energy defined similarly as $\Sigma^{R}(\mathbf{k}, \omega) = \Sigma(\mathbf{k}, i\omega_n \rightarrow \omega + i0^+)$. We note that the temperature dependence of Eq. (S33) enters implicitly in the self-energy.

D. Calculation of the pseudo-Goldstone gap

For the remainder of Sec. I, the calculation of the pseudo-Goldstone (PG) gap proceeds similarly to Ref. [4], but now with the retarded self-energy $\Sigma^{R}(\mathbf{k}, \omega)$ evaluated at finite temperature *T*. Since the PG mode appears at zero energy in linear spin-wave theory, we may expand the self energy to leading order

$$\boldsymbol{\Sigma}^{\mathrm{R}}(\boldsymbol{0},\omega) = \boldsymbol{\Sigma}^{\mathrm{R}}(\boldsymbol{0},0) + O(S^{-1}), \qquad (S34)$$

which is exact to $O(S^0)$. The subspace associated with the PG mode is two-dimensional, therefore we must calculate the poles of Eq. (S33) in the context of degenerate perturbation theory. At $O(S^0)$, this comes down to diagonalizing the effective Hamiltonian [4]

$$H_{\text{eff}} \equiv \sigma_3 \begin{pmatrix} V_0^{\dagger} \left[SM_0 + \Sigma_0^{\text{R}}(T) \right] V_0 & V_0^{\dagger} \left[SM_0 + \Sigma_0^{\text{R}}(T) \right] W_0 \\ W_0^{\dagger} \left[SM_0 + \Sigma_0^{\text{R}}(T) \right] V_0 & W_0^{\dagger} \left[SM_0 + \Sigma_0^{\text{R}}(T) \right] W_0 \end{pmatrix}$$
(S35)

projected into the zero mode subspace, where we have defined $\Sigma_0^R(T) \equiv \Sigma^R(0,0)$ to make the temperature dependence explicit. In the case of a type I mode, we may use Eq. (S27) to compute the effective Hamiltonian

$$H_{\rm eff} = \begin{pmatrix} SV_0^{\dagger} M_0 V_0 + V_0^{\dagger} \Sigma_0^{\rm R}(T) V_0 & SV_0^{\dagger} M_0 V_0 + V_0^{\dagger} \Sigma_0^{\rm R}(T) W_0 \\ -SV_0^{\dagger} M_0 V_0 - W_0^{\dagger} \Sigma_0^{\rm R}(T) V_0 & -SV_0^{\dagger} M_0 V_0 - V_0^{\dagger} \Sigma_0^{\rm R}(T) V_0 \end{pmatrix}.$$
 (S36)

Computing the eigenvalues of H_{eff} , we obtain a PG gap of $O(S^{1/2})$,

$$\Delta(T) = S^{1/2} \sqrt{V_0^{\dagger} \left[\Sigma_0^{\mathrm{R}}(T) \sigma_3 M_0 + M_0 \sigma_3 \Sigma_0^{\mathrm{R}}(T) \right] V_0}.$$
(S37)

Similarly, for a type II mode, we use Eq. (S27) to obtain

$$H_{\text{eff}} = \begin{pmatrix} V_0^{\dagger} \Sigma_0^{\text{R}}(T) V_0 & V_0^{\dagger} \Sigma_0^{\text{R}}(T) W_0 \\ -W_0^{\dagger} \Sigma_0^{\text{R}}(T) V_0 & -W_0^{\dagger} \Sigma_0^{\text{R}}(T) W_0 \end{pmatrix}.$$
 (S38)

In this case, the PG gap appears at $O(S^0)$ as

$$\Delta(T) = S^0 \sqrt{\left(\boldsymbol{V}_0^{\dagger} \boldsymbol{\Sigma}_0^{\mathsf{R}}(T) \boldsymbol{V}_0\right)^2 - \left|\boldsymbol{V}_0^{\dagger} \boldsymbol{\Sigma}_0^{\mathsf{R}}(T) \boldsymbol{W}_0\right|^2}.$$
(S39)

II. CURVATURE FORMULA

Next, we show how the PG gap computed to $O(S^0)$, at finite temperature, can be related to the curvatures of the linear spin-wave dispersion. This is in essence a nonzero temperature generalization of the curvature formula derived in Ref. [4], where it was shown that the relationship between these two quantities can be established through the first moment of the magnon spectral function, projected into the zero-mode subspace. We consider the Holstein-Primakoff expansion about a configuration that is related to the classical ground state by small rotations about the local $\hat{e}_{\alpha,x}$ and $\hat{e}_{\alpha,y}$ axes, labeled by the angles θ and ϕ respectively. Relative to the classical ground state, this transforms the local frame as

$$\hat{\boldsymbol{e}}_{\alpha,0}(\zeta) = \left(1 - |\zeta|^2\right) \hat{\boldsymbol{e}}_{\alpha,0} + \bar{\zeta} \hat{\boldsymbol{e}}_{\alpha,+} + \zeta \hat{\boldsymbol{e}}_{\alpha,-} + O(\zeta^3), \tag{S40}$$

$$\hat{\boldsymbol{e}}_{\alpha,+}(\zeta) = \left(1 - \frac{1}{2}|\zeta|^2 + \frac{1}{4}\bar{\zeta}^2 - \frac{1}{4}\zeta^2\right)\hat{\boldsymbol{e}}_{\alpha,+} - \zeta\hat{\boldsymbol{e}}_{\alpha,0} - \frac{1}{4}\zeta^2\hat{\boldsymbol{e}}_{\alpha,-} + O(\zeta^3), \tag{S41}$$

$$\hat{\boldsymbol{e}}_{\alpha,-}(\zeta) = \left(1 - \frac{1}{2}|\zeta|^2 + \frac{1}{4}\zeta^2 - \frac{1}{4}\bar{\zeta}^2\right)\hat{\boldsymbol{e}}_{\alpha,-} - \bar{\zeta}\hat{\boldsymbol{e}}_{\alpha,0} - \frac{1}{4}\bar{\zeta}^2\hat{\boldsymbol{e}}_{\alpha,+} + O(\zeta^3), \quad (S42)$$

where $\zeta \equiv (\phi + i\theta)/\sqrt{2}$. One can subsequently calculate the appropriate spin-wave theory using Eqs. (S4-S8). Alternatively, these small rotations can be related to the magnon zero-modes

$$b_0 \equiv \boldsymbol{V}_0^{\dagger} \boldsymbol{\sigma}_3 \begin{pmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_0^{\dagger} \end{pmatrix}, \qquad b_0^{\dagger} \equiv -\boldsymbol{W}_0^{\dagger} \boldsymbol{\sigma}_3 \begin{pmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_0^{\dagger} \end{pmatrix}, \qquad (S43)$$

using the Hermitian operators

$$\Phi \equiv \sqrt{\frac{SN}{2}} \left[b_0^{\dagger} + b_0 \right] + O(S^{-1/2})$$
(S44)

$$\Theta \equiv i \sqrt{\frac{SN}{2}} \left[b_0^{\dagger} - b_0 \right] + O(S^{-1/2}).$$
 (S45)

These operators generate rotations about the soft directions [4], given by the unitary operator

$$U(\phi,\theta) \equiv e^{-i\phi\Phi}e^{-i\theta\Theta}.$$
 (S46)

This can be used to define a transformed Hamiltonian that encodes these rotations $\mathcal{H}(\phi, \theta) \equiv U(\phi, \theta)^{\dagger} H U(\phi, \theta)$. Expanding to second order in ϕ and θ using the Baker–Campbell–Hausdorff formula, we obtain

$$\mathcal{H}(\phi,\theta) = H + i\theta[\Theta,H] + i\phi[\Phi,H] - \frac{1}{2}\theta^2[\Theta,[\Theta,H]] - \frac{1}{2}\phi^2[\Phi,[\Phi,H]] - \phi\theta[\Phi,[\Theta,H]] + O(\zeta^3).$$
(S47)

Proceeding to compute the curvatures of $\mathcal{H}(\phi, \theta)$, we find they are related to the nested commutators

$$\left(\frac{\partial^{2}\mathcal{H}}{\partial\phi^{2}}\right)_{0} = -\left[\Phi, \left[\Phi, H\right]\right], \qquad \left(\frac{\partial^{2}\mathcal{H}}{\partial\theta^{2}}\right)_{0} = -\left[\Theta, \left[\Theta, H\right]\right], \qquad \left(\frac{\partial^{2}\mathcal{H}}{\partial\phi\partial\theta}\right)_{0} = \left(\frac{\partial^{2}\mathcal{H}}{\partial\theta\partial\phi}\right)_{0} = -\left[\Phi, \left[\Theta, H\right]\right], \tag{S48}$$

where $(\cdots)_0$ is used as a shorthand for the evaluation at $\phi = \theta = 0$.

A. First moment of the spectral function

To relate the curvatures in Eqs. (S48) to the PG gap, we first establish a sum rule for the first moment of the magnon spectral function. More generally, the real-time retarded and advanced Green's function for a pair of operators X and Y are respectively [6]

$$G_{XY}^{\mathsf{R}}(t) \equiv -i\theta(t) \left\langle [X(t), Y(0)] \right\rangle, \tag{S49}$$

$$G_{XY}^{A}(t) \equiv i\theta(-t) \left\langle [X(t), Y(0)] \right\rangle, \tag{S50}$$

with $\theta(t)$ the Heaviside step function and the time evolution is assumed to be in the Heisenberg picture, i.e. $\frac{dX}{dt} = i[H, X]$. The corresponding spectral function is defined as

$$\mathcal{A}_{XY}(\omega) \equiv \frac{1}{2i} \int_{-\infty}^{\infty} dt \, e^{i\omega t} \left[G_{XY}^{\mathsf{R}}(t) - G_{XY}^{\mathsf{A}}(t) \right].$$
(S51)

The first moment of this spectral function is then

$$\int_{-\infty}^{\infty} d\omega \,\omega \mathcal{A}_{XY}(\omega) = \frac{1}{2} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt \, e^{i\omega t} \frac{d}{dt} \left[G_{XY}^{\mathsf{R}}(t) - G_{XY}^{\mathsf{A}}(t) \right] = -\left\langle [Y, [X, H]] \right\rangle \quad (S52)$$

The following sum rules then directly follow from Eqs. (S48)

$$\int_{-\infty}^{\infty} d\omega \,\omega \mathcal{A}_{\Phi\Phi}(\omega) = \left\langle \left(\frac{\partial^2 \mathcal{H}}{\partial \phi^2}\right)_0 \right\rangle, \qquad \int_{-\infty}^{\infty} d\omega \,\omega \mathcal{A}_{\Theta\Theta}(\omega) = \left\langle \left(\frac{\partial^2 \mathcal{H}}{\partial \theta^2}\right)_0 \right\rangle, \tag{S53}$$

$$\int_{-\infty}^{\infty} d\omega \,\omega \mathcal{A}_{\Phi\Theta}(\omega) = \left\langle \left(\frac{\partial^2 \mathcal{H}}{\partial \phi \partial \theta} \right)_0 \right\rangle, \qquad \int_{-\infty}^{\infty} d\omega \,\omega \mathcal{A}_{\Theta\Phi}(\omega) = \left\langle \left(\frac{\partial^2 \mathcal{H}}{\partial \theta \partial \phi} \right)_0 \right\rangle. \tag{S54}$$

Next, we relate these spectral functions to the single magnon spectral function, defined using Eq. (S33) as

$$\mathcal{A}(\boldsymbol{k},\omega) = \frac{1}{2i} \left[\boldsymbol{G}^{\mathrm{R}}(\boldsymbol{k},\omega) - \boldsymbol{G}^{\mathrm{R}}(\boldsymbol{k},\omega)^{\dagger} \right], \qquad (S55)$$

Projecting into the zero mode subspace, using Eq. S43, we find

$$\mathcal{A}_{0}(\omega) \equiv \begin{pmatrix} \mathcal{A}_{b_{0}b_{0}^{\dagger}}(\omega) & \mathcal{A}_{b_{0}b_{0}}(\omega) \\ \mathcal{A}_{b_{0}^{\dagger}b_{0}^{\dagger}}(\omega) & \mathcal{A}_{b_{0}^{\dagger}b_{0}}(\omega) \end{pmatrix} = \begin{pmatrix} V_{0}^{\dagger}\sigma_{3}\mathcal{A}(\mathbf{0},\omega)\sigma_{3}V_{0} & -V_{0}^{\dagger}\sigma_{3}\mathcal{A}(\mathbf{0},\omega)\sigma_{3}W_{0} \\ -W_{0}^{\dagger}\sigma_{3}\mathcal{A}(\mathbf{0},\omega)\sigma_{3}V_{0} & W_{0}^{\dagger}\sigma_{3}\mathcal{A}(\mathbf{0},\omega)\sigma_{3}W_{0} \end{pmatrix}.$$
(S56)

We can make use of Eqs. (S44-S45) and the rotated vectors

$$U_{\phi} \equiv \frac{i}{\sqrt{2}} (W_0 - V_0), \qquad U_{\theta} \equiv \frac{1}{\sqrt{2}} (W_0 + V_0),$$
 (S57)

to write the spectral functions of Φ and Θ as

$$\mathcal{A}_{\mu\nu}(\omega) = NSU^{\dagger}_{\mu}\sigma_{3}\mathcal{A}(\mathbf{0},\omega)\sigma_{3}U_{\nu}, \qquad (S58)$$

where μ , $\nu = \Phi$, Θ . The first moment of the magnon spectral functions may then be written in the compact form

$$\frac{1}{SN} \left\langle \left(\frac{\partial^2 \mathcal{H}}{\partial \lambda_{\mu} \partial \lambda_{\nu}} \right)_0 \right\rangle = U_{\mu}^{\dagger} \sigma_3 \left[\int_{-\infty}^{\infty} d\omega \, \omega \, \mathcal{A}(\mathbf{0}, \omega) \right] \sigma_3 U_{\nu}, \tag{S59}$$

where $\lambda_{\Phi} \equiv \phi$ and $\lambda_{\Theta} \equiv \theta$. This is equivalent to the result derived in [4], with the expectation value now performed at nonzero temperature *T*. It is also useful to express the Eq. (S59) in terms of the zero mode magnons explicitly. It follows from Eqs. (S44-S45) that

$$\mathcal{A}_{0}(\omega) = \frac{1}{2SN} \Big[(1+\sigma_{1})\mathcal{A}_{\Phi\Phi}(\omega) + (1-\sigma_{1})\mathcal{A}_{\Theta\Theta}(\omega) - (\sigma_{2}+i\sigma_{3})\mathcal{A}_{\Phi\Theta}(\omega) - (\sigma_{2}-i\sigma_{3})\mathcal{A}_{\Theta\Phi}(\omega) \Big].$$
(S60)

This implies that the first moment can be written as

$$\int_{-\infty}^{\infty} d\omega \,\omega \mathcal{A}_{0}(\omega) = \frac{1}{2SN} \left[(1 + \sigma_{1}) \left\langle \left(\frac{\partial^{2} \mathcal{H}}{\partial \phi^{2}} \right)_{0} \right\rangle + (1 - \sigma_{1}) \left\langle \left(\frac{\partial^{2} \mathcal{H}}{\partial \theta^{2}} \right)_{0} \right\rangle - 2\sigma_{2} \left\langle \left(\frac{\partial^{2} \mathcal{H}}{\partial \phi \partial \theta} \right)_{0} \right\rangle \right].$$
(S61)

This sum rule will enable us to demonstrate the equivalence between the PG gap and the curvature formula in the main text.

B. Calculation of the gap

Next, we discuss how to use the sum rule to calculate the PG gap for both type I and type II modes. Using Eqs. (S40-S42), we carry out a Holstein-Primakoff expansion about a rotated spin configuration to obtain $\mathcal{H}(\phi, \theta)$. First, we discuss the case of a type II PG mode, where both angles ϕ and θ correspond to soft directions. In this case, the Holstein-Primakoff expansion is

$$\mathcal{H}(\phi,\theta) = S(S+1)N\epsilon_{\rm cl} + SN\epsilon_{\rm qu}(\phi,\theta) + S\sum_{\boldsymbol{k},\alpha}\epsilon_{\boldsymbol{k},\alpha}(\phi,\theta)b_{\boldsymbol{k},\alpha}^{\dagger}b_{\boldsymbol{k},\alpha} + O(S^{1/2}),$$

as the classical configuration energy ϵ_{cl} is independent of ϕ and θ . The second derivative is then [4]

$$\frac{1}{SN} \left\langle \left(\frac{\partial^2 \mathcal{H}}{\partial \lambda_\mu \partial \lambda_\nu} \right)_0 \right\rangle = \left(\frac{\partial^2 \epsilon_{qu}}{\partial \lambda_\mu \partial \lambda_\nu} \right)_0 + \frac{1}{N} \sum_k \sum_\alpha \left(\frac{\partial^2 \epsilon_{k\alpha}}{\partial \lambda_\mu \partial \lambda_\nu} \right)_0 n_{\rm B}(\epsilon_{k,\alpha}) + O(S^{-1}), \tag{S62}$$

where $n_{\rm B}(\epsilon_{k,\alpha}) \equiv \left(\exp\left(\frac{S\epsilon_{k,\alpha}}{k_{\rm B}T}\right) - 1\right)^{-1}$ is the Bose distribution function. Eq. (S62) can be written in a more tractable form involving the free energy per spin, defined as

$$f(\phi,\theta) = S(S+1)\epsilon_{\rm cl} + S\epsilon_{\rm qu}(\phi,\theta) + \frac{k_{\rm B}T}{N}\sum_{k}\sum_{\alpha}\ln\left(1 - e^{-S\epsilon_{k,\alpha}(\phi,\theta)/k_{\rm B}T}\right).$$
 (S63)

It follows that

$$g_{\mu\nu} \equiv \frac{1}{SN} \left\langle \left(\frac{\partial^2 \mathcal{H}}{\partial \lambda_{\mu} \partial \lambda_{\nu}} \right)_0 \right\rangle = \frac{1}{S} \left[\left(\frac{\partial^2 f}{\partial \lambda_{\mu} \partial \lambda_{\nu}} \right)_0 + K_{\mu\nu} \right], \tag{S64}$$

where

$$K_{\mu\nu} \equiv \frac{S^2}{4k_{\rm B}TN} \sum_{\boldsymbol{k},\alpha} \left(\frac{\partial \epsilon_{\boldsymbol{k},\alpha}}{\partial \lambda_{\mu}} \right)_0 \left(\frac{\partial \epsilon_{\boldsymbol{k},\alpha}}{\partial \lambda_{\nu}} \right)_0 \operatorname{csch}^2 \left(\frac{S \epsilon_{\boldsymbol{k},\alpha}}{2k_{\rm B}T} \right).$$
(S65)

To calculate the PG gap, we use the fact that the first moment of the spectral function is *equivalent* to the effective Hamiltonian in Eq. (S35) (see Ref. [4] for more details). In particular, we have

$$\sigma_{3} \int_{-\infty}^{\infty} d\omega \,\omega \mathcal{A}_{0}(\omega) = H_{\text{eff}} = \frac{1}{2} \begin{pmatrix} g_{\Theta\Theta} + g_{\Phi\Phi} & g_{\Theta\Theta} - g_{\Phi\Phi} + 2ig_{\Phi\Theta} \\ g_{\Phi\Phi} - g_{\Theta\Theta} + 2ig_{\Phi\Theta} & -g_{\Theta\Theta} - g_{\Phi\Phi} \end{pmatrix}.$$
 (S66)

Diagonalizing H_{eff} , we find for the PG gap

$$\Delta(T) = S^0 \sqrt{g_{\Theta\Theta}g_{\Phi\Phi} - g_{\Phi\Theta}^2} \qquad \text{(Type II).} \tag{S67}$$

Next, we consider the case of a type I PG mode, where only one of the angles ϕ corresponds to a soft direction. In this case, the Holstein-Primakoff expansion is only stable when $\theta = 0$. Following the same line of reasoning as Ref. [4], we find

$$\boldsymbol{\sigma}_{3} \int_{-\infty}^{\infty} \mathrm{d}\omega \,\omega \mathcal{A}_{0}(\omega) = H_{\mathrm{eff}} = \frac{1}{2} \begin{pmatrix} S\left(\frac{\partial^{2}\epsilon_{\mathrm{cl}}}{\partial\theta^{2}}\right)_{0} + g_{\Phi\Phi} & S\left(\frac{\partial^{2}\epsilon_{\mathrm{cl}}}{\partial\theta^{2}}\right)_{0} - g_{\Phi\Phi} \\ g_{\Phi\Phi} - S\left(\frac{\partial^{2}\epsilon_{\mathrm{cl}}}{\partial\theta^{2}}\right)_{0} & -S\left(\frac{\partial^{2}\epsilon_{\mathrm{cl}}}{\partial\theta^{2}}\right)_{0} - g_{\Phi\Phi} \end{pmatrix}.$$
 (S68)

Diagonalizing H_{eff} , we find for the PG gap

$$\Delta(T) = S^{1/2} \sqrt{\left(\frac{\partial^2 \epsilon_{\rm cl}}{\partial \theta^2}\right)_0 g_{\Phi\Phi}} \qquad \text{(Type I).} \tag{S69}$$

C. Evaluation of $K_{\mu\nu}$

In this subsection, we discuss the evaluation of Eq. (S65) and the circumstances when this term vanishes. One can work out an explicit form of these derivatives with respect to the spin exchange matrices in a local reference frame, avoiding the need to approximate them numerically. For this,

we make use of the Hellmann-Feynmann theorem, generalized to bosonic Bogoliubov-de Gennes systems.

To begin, suppose the spin-wave theory depends on some continuous parameter λ . The linear spin-wave energies satisfy the eigenvalue equation $\sigma_3 M_k(\lambda) V_{k,\alpha}(\lambda) = \epsilon_{k,\alpha}(\lambda) V_{k,\alpha}(\lambda)$. So long as the normalization conditions Eqs. (S21-S23) are satisfied, the derivatives of the spin-wave energies satisfy

$$\frac{\partial \epsilon_{k,\alpha}}{\partial \lambda} = V_{k,\alpha}^{\dagger} \frac{\partial M_k}{\partial \lambda} V_{k,\alpha}.$$
(S70)

Making use of the local rotated frame in Eqs. (S40-S42), the first derivatives are evaluated to

$$\left(\frac{\partial \hat{\boldsymbol{e}}_{a,0}}{\partial \theta}\right)_{0} = -\frac{i}{\sqrt{2}} \left(\hat{\boldsymbol{e}}_{a,+} - \hat{\boldsymbol{e}}_{a,-}\right), \quad \left(\frac{\partial \hat{\boldsymbol{e}}_{a,+}}{\partial \theta}\right)_{0} = -\frac{i}{\sqrt{2}} \hat{\boldsymbol{e}}_{a,0}, \quad \left(\frac{\partial \hat{\boldsymbol{e}}_{a,-}}{\partial \theta}\right)_{0} = +\frac{i}{\sqrt{2}} \hat{\boldsymbol{e}}_{a,0},$$

$$\left(\frac{\partial \hat{\boldsymbol{e}}_{a,0}}{\partial \phi}\right)_{0} = +\frac{1}{\sqrt{2}} \left(\hat{\boldsymbol{e}}_{a,+} + \hat{\boldsymbol{e}}_{a,-}\right), \quad \left(\frac{\partial \hat{\boldsymbol{e}}_{a,+}}{\partial \phi}\right)_{0} = -\frac{1}{\sqrt{2}} \hat{\boldsymbol{e}}_{a,0}, \quad \left(\frac{\partial \hat{\boldsymbol{e}}_{a,-}}{\partial \phi}\right)_{0} = -\frac{1}{\sqrt{2}} \hat{\boldsymbol{e}}_{a,0}.$$

$$(S71)$$

It follows that the local exchange matrices are given by

$$\begin{pmatrix} \frac{\partial \mathcal{J}_{k,\alpha\beta}^{+-}}{\partial \theta} \\ \frac{\partial \partial \mathcal{J}_{k,\alpha\beta}^{+-}}{\partial \theta} \end{pmatrix}_{0} = +\frac{i}{\sqrt{2}} \left(\mathcal{J}_{k,\alpha\beta}^{+0} - \bar{\mathcal{J}}_{k,\beta\alpha}^{+0} \right), \qquad \left(\frac{\partial \mathcal{J}_{k,\alpha\beta}^{++}}{\partial \theta} \right)_{0} = -\frac{i}{\sqrt{2}} \left(\mathcal{J}_{k,\alpha\beta}^{+0} + \mathcal{J}_{-k,\beta\alpha}^{+0} \right), \\ \left(\frac{\partial \mathcal{J}_{k,\alpha\beta}^{+-}}{\partial \phi} \right)_{0} = -\frac{1}{\sqrt{2}} \left(\mathcal{J}_{k,\alpha\beta}^{+0} + \bar{\mathcal{J}}_{k,\beta\alpha}^{+0} \right), \qquad \left(\frac{\partial \mathcal{J}_{k,\alpha\beta}^{++}}{\partial \phi} \right)_{0} = -\frac{1}{\sqrt{2}} \left(\mathcal{J}_{k,\alpha\beta}^{+0} + \mathcal{J}_{-k,\beta\alpha}^{+0} \right),$$
(S72)

where we have used the properties $\mathcal{J}_{k,\alpha\beta}^{0-} = \bar{\mathcal{J}}_{k,\beta\alpha}^{+0}$ and $\mathcal{J}_{k,\alpha\beta}^{0+} = \mathcal{J}_{-k,\beta\alpha}^{+0}$. We may then evaluate the derivatives of the spin-wave energies using Eqs. (S70,S72) as

$$\left(\frac{\partial \epsilon_{\boldsymbol{k},\alpha}}{\partial \lambda_{\mu}}\right)_{0} = V_{\boldsymbol{k},\alpha}^{\dagger} \begin{pmatrix} \boldsymbol{C}_{\boldsymbol{k},\mu} & \boldsymbol{D}_{\boldsymbol{k},\mu} \\ \boldsymbol{D}_{\boldsymbol{k},\mu}^{\dagger} & \boldsymbol{C}_{-\boldsymbol{k},\mu}^{\dagger} \end{pmatrix} V_{\boldsymbol{k},\alpha},$$
(S73)

where

$$\left[\boldsymbol{C}_{\boldsymbol{k},\mu}\right]_{\alpha\beta} \equiv \left(\frac{\partial \mathcal{J}_{\boldsymbol{k},\alpha\beta}^{+-}}{\partial \lambda_{\mu}}\right)_{0}, \quad \left[\boldsymbol{D}_{\boldsymbol{k},\mu}\right]_{\alpha\beta} \equiv \left(\frac{\partial \mathcal{J}_{\boldsymbol{k},\alpha\beta}^{++}}{\partial \lambda_{\mu}}\right)_{0}.$$
 (S74)

The $K_{\mu\nu}$ term may then be evaluated as

$$K_{\mu\nu}(T) = \frac{S^2}{4TN} \sum_{\boldsymbol{k},\alpha} V_{\boldsymbol{k},\alpha}^{\dagger} \begin{pmatrix} \boldsymbol{C}_{\boldsymbol{k},\mu} & \boldsymbol{D}_{\boldsymbol{k},\mu} \\ \boldsymbol{D}_{\boldsymbol{k},\mu}^{\dagger} & \boldsymbol{C}_{-\boldsymbol{k},\mu}^{\dagger} \end{pmatrix} V_{\boldsymbol{k},\alpha} V_{\boldsymbol{k},\alpha}^{\dagger} \begin{pmatrix} \boldsymbol{C}_{\boldsymbol{k},\nu} & \boldsymbol{D}_{\boldsymbol{k},\nu} \\ \boldsymbol{D}_{\boldsymbol{k},\nu}^{\dagger} & \boldsymbol{C}_{-\boldsymbol{k},\nu}^{\dagger} \end{pmatrix} V_{\boldsymbol{k},\alpha} \operatorname{csch}^2 \left(\frac{S\epsilon_{\boldsymbol{k},\alpha}}{2T} \right).$$
(S75)

This expression vanishes in the zero-temperature limit, that is $K_{\mu\nu} \rightarrow 0$ as $T \rightarrow 0^+$. The expression in Eq. (S75) is enormously useful as it involves only the linear spin-wave eigenvectors and exchange

matrices, and therefore circumvents the need to calculate derivatives of the spin-wave energies numerically (e.g. using a finite difference method).

We note the $\mathcal{J}_{k,\alpha\beta}^{+0}$ exchange couplings appearing in Eq. (S72) are directly related to the three-Magnon interaction vertices in Eq. (S7), given by

$$T_{\boldsymbol{k}\boldsymbol{k}'}^{\alpha\beta\mu} = -\delta_{\alpha\mu}\mathcal{J}_{\boldsymbol{k}',\beta\alpha}^{+0} - \delta_{\beta\mu}\mathcal{J}_{\boldsymbol{k},\alpha\beta}^{+0}.$$
(S76)

We may therefore conclude that if the spin-wave theory does not have any three-magnon, interactions, i.e. $\mathcal{T}_{kk'}^{\alpha\beta\mu} = 0$, then $K_{\mu\nu}(T) = 0$ and the PG gap can be calculated to $O(S^0)$ from the curvature of the linear spin-wave free energy. In the general case where three-body interactions are present, this additional term $K_{\mu\nu}(T)$ will be nonzero, however we emphasize that it still allows one to compute the PG gap within the framework of linear spin-wave theory.

III. MODELS

We now discuss details of two additional magnetic pyrochlore materials, where the quantum spin models are known to exhibit ObD.

A. Yb₂Ge₂O₇

First, we discuss the XY pyrochlore antiferromagnet $Yb_2Ge_2O_7$. Similar to the material $Er_2Ti_2O_7$ discussed in the main text, strong spin-orbit effects lead to a highly anisotropic (pseudo) spin- $\frac{1}{2}$ model

$$H = \sum_{\langle i,j \rangle} \left[J_{zz} S_i^z S_j^z - J_{\pm} \left(S_i^+ S_j^- + S_i^- S_j^+ \right) + J_{\pm\pm} \left(\gamma_{ij} S_i^+ S_j^+ + \text{H.c.} \right) + J_{z\pm} \left(\zeta_{ij} \left[S_i^z S_j^+ + S_i^+ S_j^z \right] + \text{H.c.} \right) \right],$$
(S77)

where γ_{ij} , ζ_{ij} are bond-dependent phase factors (see Ref. [8]). The four nearest-neighbor couplings have been fitted to inelastic neutron scattering data, with the best fit given by [9]

$$J_{zz} = 0.128 \text{ meV}, \ J_{\pm} = 0.138 \text{ meV}, \ J_{\pm\pm} = 0.044 \text{ meV}, \ J_{z\pm} = -0.188 \text{ meV}.$$
 (S78)

The classical ground states are non-collinear antiferromagnetic configurations of spins lying in the local XY planes perpendicular to the local [111] cubic axes of the pyrochlore lattice [8], parametrized by an accidental U(1) degeneracy [10]. Below $T_c \approx 0.572$ K [9], ObD selects one of the six " ψ_3 " states, leading to a type I PG mode.



FIG. S1. PG gap for the pyrochlore antiferromagnet Yb₂Ge₂O₇, calculated using the exchange couplings from Ref. [9]. The classical ground states are parametrized by an accidental U(1) degeneracy, corresponding to a type I mode. The gap is plotted up to $T_c/2 = 286$ mK.

The temperature dependence of PG gap for Yb₂Ge₂O₇ is depicted in Fig. S1 over a range of temperatures up to $T_c/2$. We find a zero-temperature contribution to the gap of $\Delta_0 = 13.028 \ \mu eV$, and a very small thermal correction of $\Delta(T_c/2) - \Delta_0 \approx 6$ neV. At low-temperature, the gap scales proportional to T^4 as expected for a type I mode.

B. $Lu_2V_2O_7$

Finally, we discuss the somewhat unique scenario of the pyrochlore ferromagnet $Lu_2V_2O_7$. In this material, the V^{4+} transition metal ions are magnetic, leading to the effective spin- $\frac{1}{2}$ model

$$H = -J \sum_{\langle i,j \rangle} S_i \cdot S_j - \sum_{\langle i,j \rangle} D_{ij} \cdot (S_i \times S_j), \qquad (S79)$$

where the DM vectors are of the "indirect" type [12]. The two nearest-neighbor couplings have been fitted to inelastic neutron scattering data, with the best fit given by [11]

$$J = 8.22 \text{ meV}, \qquad |D_{ij}| = 1.5 \text{ meV}.$$
 (S80)



FIG. S2. PG gap for the pyrochlore ferromagnet Lu₂V₂O₇, calculated using the exchange couplings from Ref. [11]. The ground states are parametrized by an accidental O(3) degeneracy, corresponding to a type II mode. The gap is plotted up to $T_c/2 = 36.5$ K.

The ground states are collinear ferromagnetic configurations parametrized by an accidental O(3) degeneracy [13]. Below $T_c \approx 73$ K [11], ObD (within the model of Eq. (S79) selects one of the $\langle 111 \rangle$ directions for the bulk magnetization [13], leading to a type II PG mode.

The temperature dependence of the PG gap for Lu₂V₂O₇ is depicted in Fig. S2 over a range of temperatures up to $T_c/2$. In this case, there is no zero-temperature contribution to the gap, consistent with recent work in Ref. [13] arguing that this material exhibits ObD *without* quantum zero-point fluctuations. Moreover, we find at low-temperature a gap scales proportional to $T^{7/2}$, distinct from the expected $T^{5/2}$ scaling for a type II mode. This exception arises due to the observation pointed out in Ref. [13], that the anisotropy enters the spin-wave dispersion at $O(k^4)$ in the wavevector, when $|D_{ij}| \ll J$. This implies the linear spin-wave free energy, at low-temperature, takes the form

$$f(\phi, \theta) = aT^{5/2} + b(\phi, \theta)T^{7/2} + O(T^{9/2}),$$
(S81)

with the coefficient *a* being independent of the spin orientation. The $K_{\mu\nu}$ terms entering the curvature formula will have a similar low-temperature expansion, and Eq. (S67) is consistent with

this power-law scaling.

- [1] J.-P. Blaizot and G. Ripka, Quantum Theory of Finite Systems (MIT Press, 1986).
- [2] J. Colpa, Diagonalization of the quadratic boson hamiltonian, Phys. A: Stat. Mech. Appl. 93, 327 (1978).
- [3] H. Kondo, Y. Akagi, and H. Katsura, Non-Hermiticity and topological invariants of magnon Bogoliubov–de Gennes systems, Prog. Theor. Exp. Phys 2020, 12A104 (2020).
- [4] J. G. Rau, P. A. McClarty, and R. Moessner, Pseudo-Goldstone Gaps and Order-by-Quantum Disorder in Frustrated Magnets, Phys. Rev. Lett. 121, 237201 (2018).
- [5] E. Rastelli, Statistical Mechanics of Magnetic Excitations: from Spin Waves to Stripes to Checkerboards (World Scientific, London, 2013).
- [6] G. D. Mahan, Many-Particle Physics, 3rd ed. (Springer New York, 2000).
- [7] A. Altland and B. Simons, *Condensed Matter Field Theory*, 2nd ed. (Cambridge University Press, 2010).
- [8] J. G. Rau and M. J. P. Gingras, Frustrated Quantum Rare-Earth Pyrochlores, Annual Review of Condensed Matter Physics 10, 357 (2019).
- [9] C. L. Sarkis, J. G. Rau, L. D. Sanjeewa, M. Powell, J. Kolis, J. Marbey, S. Hill, J. A. Rodriguez-Rivera, H. S. Nair, D. R. Yahne, S. Säubert, M. J. P. Gingras, and K. A. Ross, Unravelling competing microscopic interactions at a phase boundary: A single-crystal study of the metastable antiferromagnetic pyrochlore Yb₂Ge₂O₇, Phys. Rev. B **102**, 134418 (2020).
- [10] H. Yan, O. Benton, L. Jaubert, and N. Shannon, Theory of multiple-phase competition in pyrochlore magnets with anisotropic exchange with application to Yb₂Ti₂O₇, Er₂Ti₂O₇, and Er₂Sn₂O₇, Phys. Rev. B **95**, 094422 (2017).
- [11] M. Mena, R. S. Perry, T. G. Perring, M. D. Le, S. Guerrero, M. Storni, D. T. Adroja, C. Rüegg, and D. F. McMorrow, Spin-Wave Spectrum of the Quantum Ferromagnet on the Pyrochlore Lattice Lu₂V₂O₇, Phys. Rev. Lett. **113**, 047202 (2014).
- [12] M. Elhajal, B. Canals, R. Sunyer, and C. Lacroix, Ordering in the pyrochlore antiferromagnet due to Dzyaloshinsky-Moriya interactions, Phys. Rev. B 71, 094420 (2005).
- [13] A. Hickey, D. Lozano-Gómez, and M. J. P. Gingras, Order-by-disorder without quantum zero-point fluctuations in the pyrochlore Heisenberg ferromagnet with Dzyaloshinskii-Moriya interactions, Phys.

Rev. B 111, 184434 (2025).