

# Spin Space Symmetry Without Spin Space Groups

Jeffrey G. Rau<sup>1,\*</sup>

<sup>1</sup>*Department of Physics, University of Windsor, 401 Sunset Avenue, Windsor, Ontario, N9B 3P4, Canada*  
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Spin space groups provide a complete symmetry classification of magnetic structures in the non-relativistic limit and have become an increasingly useful language for organizing magnetic phases and their physical properties. Each spin space group with a given parent space group is specified, up to spin-only symmetries, by a real representation of that space group acting in spin space. We argue that working directly with this representation not only avoids the construction and identification of the group, but also organizes and simplifies the resulting symmetry constraints on allowed responses. Determining the presence or absence of magnetic compensation, piezomagnetism, spin and charge transport, the Edelstein effect, and momentum-space spin textures becomes an ordinary problem in the representation theory of space groups. The resulting conditions reduce to overlaps between the three exterior powers of the defining representation and multipole representations of the point group. Several physically distinct responses then reduce to the same condition on the defining representation. For example, a  $p$ -wave spin texture and the time-even Edelstein effect are allowed under the same condition, as are piezomagnetism and a  $d$ -wave spin texture in compensated order. The wave-vector content of the exterior powers gives a further hierarchy of selection rules: depending on the relevant power, a uniform response requires one, two, or three ordering wave vectors to sum to a reciprocal lattice vector. These rules can exclude many responses before point-group symmetry is considered. We illustrate the approach with examples drawn from a wide variety of materials hosting collinear, coplanar and non-coplanar magnetic order. Finally, we discuss extensions of these results to magnetic space groups, connections to Landau theories of collinear order, and the characterization of odd-parity magnets.

## I. INTRODUCTION

Symmetry classification is one of the basic organizing principles of condensed-matter physics. For crystalline systems, space groups provide a finite atlas of possible spatial symmetries [1, 2]; once a material has been assigned to a particular group, the corresponding symmetry constraints follow independent of the microscopic details. In the presence of magnetic ordering, this group structure can be generalized to include symmetries involving not only spatial operations but also spatial operations combined with operations that act in spin space [3].

How spatial and spin operations combine depends on the strength of relativistic effects. The magnetic space groups [2, 3] assume spin and spatial rotations are locked together, as required when spin-orbit coupling or dipolar interactions are significant. In the opposite, non-relativistic limit, spin rotations decouple from spatial rotations entirely, and the larger classes of spin space groups [4, 5] emerge. Recent work has revived focus on this limit, partly motivated by the physics of altermagnetism [6–8], culminating in a complete classification and enumeration of the possible spin space groups [9–11].

There is, however, a distinction between these two problems. First is the *classification problem* [9–11]: determine all possible symmetry groups, identify when two groups are equivalent, and choose a canonical representative and label for each equivalence class. Second is the *inference problem*: given the symmetry of a particular ordered state, determine which observables, couplings, degeneracies, and response functions are allowed or forbidden. A complete classification provides

one route to the second problem: once a state has been assigned its group, the allowed form of each quantity follows by imposing invariance under each of its elements in turn, as in Neumann’s principle. For response tensors this inference has been developed systematically using spin-group transformation laws [12, 13] and implemented in automated software tools [14]. The question addressed here is how to organize these constraints directly through the representation of the space group, acting in spin space, that specifies the spin space group, making their common spin-space structure and dependence on translations explicit.

For spin space groups this observation can be made particularly transparent. In the non-relativistic limit spin rotations and time reversal act independently of the lattice, and for real quantities they combine into  $O(3)$ , so the paramagnetic phase of a crystal with space group  $G_0$  has symmetry  $O(3) \times G_0$ . A magnetically ordered state has a spin space group  $SSG \leq O(3) \times G$ , where the parent space group  $G \leq G_0$  of the order may be smaller than that of the crystal. For non-coplanar order its elements are spin-space operations  $[\Gamma(g)||g]$ , one for each  $g \in G$ , and the group structure makes the matrices  $\Gamma(g)$  a real representation,  $\Gamma$ , of  $G$ , which we call the defining representation (Section II). The classification of spin space groups can be formulated as the problem of enumerating all such inequivalent representations for all parent space groups [9, 11]. Up to an overall choice of spin frame, this representation contains precisely the information required to determine how spin-dependent quantities transform. Collinear, coplanar, and non-coplanar states are described naturally by real representations of dimension one, two, and three [9], respectively, with collinear and coplanar orderings enjoying additional spin-only symmetries.

One route to obtaining symmetry constraints on physical

\* jrau@uwindsor.ca

quantities may therefore be sketched as

$$\begin{aligned}
 &\text{Parent Space Group } (G) + \text{Representation } (\Gamma), \\
 &\quad \downarrow \\
 &\text{Spin Space Group } (\text{SSG} \leq O(3) \times G), \\
 &\quad \downarrow \\
 &\text{Physical Consequences } (\mathbf{M}, \mathbf{C}_{\mu\nu}, \sigma^-, \dots),
 \end{aligned} \tag{1}$$

where the first step constructs the spin space group, which may then be identified within a classification, and the second applies its elements to each quantity of interest. The central point of this work is that the second step can be framed directly in terms of  $\Gamma$ . Doing so avoids the construction and identification of the group, and has the further advantage that it organizes the resulting constraints: the transformation laws of many observables are fixed by a few objects built from  $\Gamma$ , so that the constraints on different observables are related, and their origin in the ordered state is visible.

The relevant objects are the exterior powers  $\Lambda^1\Gamma$ ,  $\Lambda^2\Gamma$  and  $\Lambda^3\Gamma$  of the defining representation. For the quantities in Table I, the spin transformation law is fixed by the presence or absence of a spin index and by time-reversal parity. A time-odd spin vector transforms under  $\Lambda^1\Gamma$ , a time-even spin vector under  $\Lambda^2\Gamma$ , and a time-odd spin scalar under  $\Lambda^3\Gamma$ , with the constraints of the spin-only symmetries included for collinear and coplanar order. The spatial tensor character then determines which multipole representation the selected power must overlap. Consequently, observables selecting the same power share constraints, and a power that vanishes, as  $\Lambda^2\Gamma$  and  $\Lambda^3\Gamma$  do for collinear order, removes all of them at once.

An important consequence is a hierarchy of selection rules arising from translational symmetry, independent of the detailed point-group representation content. For a three-dimensional representation, a spatially uniform response controlled by  $\Lambda^1\Gamma$  requires an ordering wave vector at zero, one controlled by  $\Lambda^2\Gamma$  requires two wave vectors that sum to zero, and one controlled by  $\Lambda^3\Gamma$  requires all three to sum to zero, in each case modulo a reciprocal lattice vector (Fig. 1). The symmetry analysis therefore proceeds in two steps: first identifying whether the relevant exterior power contains a translation-invariant sector, and then determining whether that sector overlaps the spatial representation of the observable. In some cases the first step alone is enough to rule out a response. For example, a non-coplanar triple- $Q$  state with three nonzero ordering wave vectors whose sum vanishes can support an anomalous Hall response even when the spin conductivities and momentum-space spin texture are forbidden. We demonstrate this separation explicitly for the tetrahedral order in  $\text{CoTa}_3\text{S}_6$  in Section V E.

Applied to the net spin magnetization, piezomagnetism [15–17], spin and charge transport [18, 19], momentum-space spin textures [6, 20–22], and the Edelstein effect [23], the exterior powers give the conditions summarized in Table I. Several responses share identical symmetry criteria: for compensated order, piezomagnetism and a  $d$ -wave spin texture are allowed under the same conditions, as are a  $p$ -wave spin texture and the time-even Edelstein effect. Relations of this kind are not

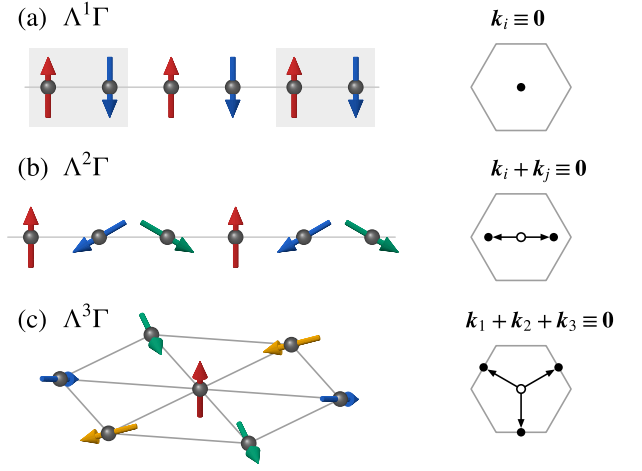


FIG. 1. Orders whose wave vectors give  $\Lambda^p\Gamma$  a uniform component, Eq. (13a), each beside its wave vectors in the Brillouin zone. (a) A two-sublattice order at  $\mathbf{k} = \mathbf{0}$ , the same in every unit cell (shaded). (b) A  $120^\circ$  spiral, with wave vectors  $\pm\mathbf{k}$ . (c) A tetrahedral triple- $Q$  order, with wave vectors at the three  $M$  points.

| Response                    |           | Symbol                   | Required overlap                                                        | Section |
|-----------------------------|-----------|--------------------------|-------------------------------------------------------------------------|---------|
| Net spin magnetization      |           | $\mathbf{M}$             | $\Gamma \cap \Gamma_0^+$                                                | IV A    |
| Piezomagnetism              |           | $\mathbf{C}_{\mu\nu}$    | $\Gamma \cap \Gamma_2^+$                                                | IV B    |
| Spin conductivity           | Time-odd  | $\Sigma^-$               | $\Gamma \cap (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+)$          | IV C 1  |
|                             | Time-even | $\Sigma^+$               | $\Lambda^2\Gamma \cap (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+)$ | IV C 1  |
| Anomalous Hall conductivity |           | $\sigma^-$               | $\Lambda^3\Gamma \cap \Gamma_1^+$                                       | IV C 2  |
| Edelstein effect            | Time-odd  | $\alpha^-$               | $\Gamma \cap \Gamma_1^-$                                                | IV E    |
|                             | Time-even | $\alpha^+$               | $\Lambda^2\Gamma \cap \Gamma_1^-$                                       | IV E    |
| Spin texture                | Even rank | $\mathbf{s}(\mathbf{q})$ | $\Gamma \cap \Gamma_\ell^+$                                             | IV D    |
|                             | Odd rank  | $\mathbf{s}(\mathbf{q})$ | $\Lambda^2\Gamma \cap \Gamma_\ell^-$                                    | IV D    |

TABLE I. Summary of the symmetry conditions for the non-relativistic responses derived in this work. Each is expressed as a nonzero overlap of an exterior power of the defining representation  $\Gamma$  with a multipole representation  $\Gamma_\ell^\pm$  of the parent space group  $G$ . The section in which each condition is derived is indicated. For collinear order  $\Lambda^2\Gamma = \Lambda^3\Gamma = 0$ , and for coplanar order  $\Lambda^3\Gamma = 0$ , so the responses selecting these powers vanish. The piezomagnetic condition assumes compensated order, and a rank- $\ell$  contribution to the spin texture requires the overlap shown for  $\ell$  of the stated parity.

as apparent when each response is constrained separately by the elements of the group.

We stress that our goal is not to replace spin space groups, nor to argue that their classification is unnecessary. A group classification remains essential when one wishes to enumerate possible symmetry types, compare structures across different crystallographic settings, construct databases [24–27], or assign canonical symmetry labels [28]. The full spin space group is also the natural object for questions about individual states, such as band degeneracies, band connectivity and symmetry-diagnosed topology, rather than the allowed form of a response (Section V E). Imposed element by element, the spin space group gives the same allowed forms. Organizing

these constraints through the defining representation makes their common origin explicit.

Beyond the general methods noted above, several individual results have already been reported. Watanabe *et al.* [12] analyzed the spin-group conditions for the non-relativistic Hall response, and noted that in  $\text{CoTa}_3\text{S}_6$  it occurs without non-relativistic spin–momentum locking. The symmetry equivalence of piezomagnetism and a  $d$ -wave spin texture was discussed for magnetic point groups by Radaelli [29], and the spin-texture parity relations appear in the classification of Song *et al.* [22].

Working with the exterior powers, more general connections can be exposed systematically. We use the exterior powers to derive a wave-vector selection hierarchy of Section III, which can settle a response before any point-group character is computed. The same construction restricts the non-relativistic anomalous Hall response to non-cubic parent point groups and, for compensated order at  $\mathbf{k} = 0$ , to defining representations built from three real one-dimensional irreps (Section IV C 2). We also characterize odd-parity magnets systematically, fixing their wave-vector content, inversion parity and wave character directly from  $\Gamma$  (Section VI C). For example, an interior non-TRIM pair  $\{\mathbf{k}, -\mathbf{k}\}$  in coplanar order always gives a  $p$ -wave texture and a time-even Edelstein response. Combining the representation conditions with Onsager reciprocity, we further show that any non-relativistic time-odd spin conductivity in the collinear orders of  $\alpha\text{-Fe}_2\text{O}_3$  or  $\text{CuFeS}_2$  is antisymmetric in its spatial indices and requires interactions coupling the spin channels (Section IV C 1).

The outline of the paper is as follows. We first review in Section II how collinear, coplanar, and non-coplanar spin-space symmetries are encoded by real representations of the parent space group. Section III then collects the properties of the exterior powers of that representation, and of their wave-vector content, that are used throughout, and Section IV derives from them the conditions of Table I. We next apply these in Section V to eight materials spanning collinear, coplanar and non-coplanar order at both zero and finite ordering wave vector, including two,  $\text{MnO}$  and  $\text{Gd}_3(\text{Ru}_{0.95}\text{Rh}_{0.05})_4\text{Al}_{12}$ , in which the order selects part of a wave-vector star, so that its parent space group is smaller than the crystallographic one, and several in which the dimension and wave-vector content of  $\Gamma$  alone settle every response. We close with a discussion of how the same construction relates to conventional magnetic space groups, each of which is likewise encoded in a real one-dimensional representation of its parent space group (Section VI A), to the Landau theory of collinear magnetic transitions (Section VI B), to odd-parity magnets (Section VI C), and to existing spin space group classifications of the momentum-space spin textures (Section VI D).

## II. DEFINITIONS

Consider a crystal with space group  $G_0$  and full spin rotation symmetry,  $\mathbb{Z}_2^T \times \text{SO}(3) \times G_0$ , where  $\mathbb{Z}_2^T$  is time reversal. Since we work exclusively with real quantities, time reversal and spin

rotations may be combined as

$$\mathbb{Z}_2^T \times \text{SO}(3) \rightarrow \text{O}(3), \quad (2)$$

interpreting any  $U \in \text{O}(3)$  with  $\det U = -1$  as a spin rotation followed by time reversal, which for real quantities is a spin inversion. The paramagnetic symmetry group is then  $\text{O}(3) \times G_0$  where  $G_0$  is the crystal space group.

At a magnetic phase transition this is broken to a spin space group  $\text{SSG} \leq \text{O}(3) \times G$  where  $G \leq G_0$  is the parent space group of the magnetic order. This parent space group may be smaller than the space group of the crystal: it contains those spatial operations that preserve the magnetic ordering pattern up to a global transformation in spin space. An explicit method for construction of  $G$  and  $\Gamma$  for a commensurate magnetic structure is given in Appendix C. The elements of the spin space group may be written as the spin-space operations

$$[\Gamma(g)\|g], \quad (3)$$

with  $g \in G$  and  $\Gamma(g) \in \text{O}(3)$ . For non-coplanar order, the group structure of SSG requires  $\Gamma(g_1)\Gamma(g_2) = \Gamma(g_1g_2)$  for all  $g_1, g_2 \in G$ , so the  $\Gamma$  form a real three-dimensional representation of the parent space group  $G$  [4, 5, 9], which need not be irreducible. A choice of basis for this representation fixes a (right-handed) frame in spin space: a global spin rotation  $\tilde{\Gamma}(g) \equiv \mathbf{R}\Gamma(g)\mathbf{R}^{-1}$ , with  $\mathbf{R} \in \text{SO}(3)$  arbitrary, yields an equivalent subgroup. We treat spin space groups related by such a rotation as equivalent. If the order is collinear or coplanar, the group also contains the elements of the spin-only group.

The representation  $\Gamma$  is a representation of the full space group. Its action on lattice translations encodes the wave-vector content of the magnetic order. In particular, the relevant wave vectors can be organized into stars under the point-group action: if  $\mathbf{k}$  occurs in the magnetic structure, then symmetry relates it to the other wave vectors in the star of  $\mathbf{k}$ , modulo reciprocal lattice vectors. The representation  $\Gamma$  therefore reflects not only the propagation wave vector itself, but also how the different arms of its star are related by the spatial symmetries (not just a little group representation).

For collinear and coplanar orders the magnetic structure spans only a one- or two-dimensional subspace of spin space, and after factoring out the additional spin-only symmetries the corresponding defining representation is one- or two-dimensional [9]. The remaining spin directions are acted on by additional spin-only symmetries. For collinear order the representation is a real one-dimensional irrep,  $\Gamma(g) = \pm 1$ . Denoting the collinear axis by  $\hat{N}$ , convenient three-dimensional representatives of the full spin-space operations can be chosen to be of the form

$$[\Gamma(g)\mathbf{1}\|g], \quad (4)$$

where  $\Gamma(g) = +1$  has no spin operation and  $\Gamma(g) = -1$  indicates time-reversal. The group is extended to include all rotations  $\mathbf{R}_{\hat{N}}(\theta)$  about the collinear direction, together with the two-fold rotations perpendicular to  $\hat{N}$  followed by time reversal. [30]

For coplanar order the representation is a real two-dimensional representation  $\Gamma$ . Taking the preferred plane, spanned by  $\hat{N}_1$  and  $\hat{N}_2$ , perpendicular to  $\hat{z}$ , convenient three-dimensional representatives of the full spin-space operations can be written

$$\left[ \begin{pmatrix} \Gamma(g) & 0 \\ 0 & \det \Gamma(g) \end{pmatrix} \right] g. \quad (5)$$

The group is again extended to include spin-only operations, here by a two-fold rotation  $\mathbf{R}_z(\pi)$  combined with time reversal [31].

Groups obtained this way are not necessarily distinct. To carry out a classification we would further need to remove groups that are equivalent under changes of axes or origin, so-called affine equivalences [1]. This is the program carried out in Xiao *et al.* [9] and in the companion enumerations of Chen *et al.* [10] and Jiang *et al.* [11] using different techniques.[32]

Note that magnetic orders whose spin space group has a parent space group identical to the underlying crystal space group ( $G = G_0$ ) are called *regular magnetic orders* [33]. These orders have the property that they are stationary points of Heisenberg spin Hamiltonians (absent free continuous parameters in the ansatz). Messio *et al.* [33] used an algebraic approach to classify all spin wallpaper groups with parent groups of  $p6m$  or  $p4m$  symmetry, and constructed examples of the associated regular magnetic orders.

### III. EXTERIOR POWERS

Before turning to specific observables it will be helpful to discuss some properties of the defining representation that are used repeatedly below. Let  $V$  denote the spin space on which the real representation  $\Gamma$  acts. The natural representations on antisymmetric products of spin vectors are the exterior powers  $\Lambda^p \Gamma$  acting on  $\Lambda^p V$  [2, 34]. Here  $\Lambda^p V$  is the space of totally antisymmetric  $p$ -fold products, spanned by the wedge products  $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p$ , which change sign under the exchange of any two factors; it has dimension  $\binom{n}{p}$  for  $\dim V = n$  and vanishes for  $p > n$ . Acting with  $\Gamma(g)$  on each factor preserves the antisymmetry and so defines a representation,

$$\Lambda^p \Gamma(g) (\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p) \equiv (\Gamma(g)\mathbf{v}_1) \wedge \cdots \wedge (\Gamma(g)\mathbf{v}_p). \quad (6)$$

For non-coplanar order, where  $\Gamma$  is three-dimensional, these satisfy

$$\Lambda^1 \Gamma \equiv \Gamma, \quad (7a)$$

$$\Lambda^2 \Gamma \equiv \Gamma \otimes \det \Gamma, \quad (7b)$$

$$\Lambda^3 \Gamma \equiv \det \Gamma. \quad (7c)$$

The third holds because  $\Gamma(g)$  rescales the oriented volume element spanning the one-dimensional  $\Lambda^3 V$  by its determinant. The second isomorphism follows from  $\Lambda^2 V \equiv V^* \otimes \Lambda^3 V$  together with the identification  $V^* \equiv V$ ; this is a generalization of the familiar identification of planes in three dimensions with axial vectors.

Geometrically, these correspond to

$$\begin{aligned} \Lambda^1 \Gamma &: \text{spin vector,} \\ \Lambda^2 \Gamma &: \text{spin pseudovector,} \\ \Lambda^3 \Gamma &: \text{spin pseudoscalar.} \end{aligned}$$

The second can be seen from the cross product. For any  $\Gamma(g) \in O(3)$ ,

$$(\Gamma(g)\mathbf{S}_1) \times (\Gamma(g)\mathbf{S}_2) = \det \Gamma(g) \Gamma(g)(\mathbf{S}_1 \times \mathbf{S}_2), \quad (8)$$

so an antisymmetric spin bilinear, such as the vector spin chirality  $\mathbf{S}_1 \times \mathbf{S}_2$  of two moments, transforms under  $\Lambda^2 \Gamma$  and, unlike a spin vector, is even under spin inversion. The third is the oriented volume in spin space, realized most simply by the scalar spin chirality

$$\chi_{123} = \mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3), \quad (9)$$

which transforms as

$$\chi_{123} \rightarrow \det \Gamma(g) \chi_{123}. \quad (10)$$

Both reappear in Section IV C, in the time-even spin conductivity and the anomalous Hall conductivity.

For a two-dimensional representation  $\Gamma$  the exterior powers are

$$\Lambda^1 \Gamma \equiv \Gamma, \quad (11a)$$

$$\Lambda^2 \Gamma \equiv \det \Gamma, \quad (11b)$$

where  $\Lambda^3 \Gamma = 0$ . Here  $\Lambda^2 \Gamma = \det \Gamma$  is the oriented area of the spin plane or, equivalently, the representation carried by its normal  $\hat{N}_1 \times \hat{N}_2$ . For a one-dimensional representation  $\Gamma$  the higher exterior powers vanish, with

$$\Lambda^1 \Gamma \equiv \Gamma, \quad (12)$$

and  $\Lambda^2 \Gamma = \Lambda^3 \Gamma = 0$ . For an irreducible  $\Gamma$  of dimension two or three,  $\det \Gamma$  can be read from Table II. For a reducible  $\Gamma$  the exterior powers follow from those of its irreducible summands, as given in Appendix A.

We can make general statements about the wave-vector content of these exterior powers. Restricted to the translations, a real  $n$ -dimensional representation  $\Gamma$  becomes, over  $\mathbb{C}$ , a sum of characters  $[\mathbf{1}|\mathbf{t}] \mapsto e^{i\mathbf{k}_i \cdot \mathbf{t}}$ , and we call  $\{\mathbf{k}_1, \dots, \mathbf{k}_n\}$  its wave-vector content. Wedge products of translation eigenvectors diagonalize the translations on  $\Lambda^p V$ , with characters  $e^{i(\mathbf{k}_{i_1} + \cdots + \mathbf{k}_{i_p}) \cdot \mathbf{t}}$  for  $i_1 < \cdots < i_p$ , so  $\Lambda^p \Gamma$  contains a component at  $\mathbf{k} = 0$  if and only if

$$\Lambda^1 \Gamma : \mathbf{k}_i \equiv \mathbf{0} \text{ for some } i, \quad (13a)$$

$$\Lambda^2 \Gamma : \mathbf{k}_i + \mathbf{k}_j \equiv \mathbf{0} \text{ for some } i \neq j, \quad (13b)$$

$$\Lambda^3 \Gamma : \sum_i \mathbf{k}_i \equiv \mathbf{0}, \quad (13c)$$

in each case modulo a reciprocal lattice vector. The three exterior powers therefore test whether one ordering wave vector vanishes, whether two cancel, or whether all three sum to

| $P$      | $\Gamma \rightarrow \det \Gamma$ | $P$      | $\Gamma \rightarrow \det \Gamma$                    |
|----------|----------------------------------|----------|-----------------------------------------------------|
| $C_3$    | $E \rightarrow A$                | $C_6$    | $E_1, E_2 \rightarrow A$                            |
| $S_6$    | $E_g, E_u \rightarrow A_g$       | $C_{6h}$ | $E_{1g}, E_{2g}, E_{1u}, E_{2u} \rightarrow A_g$    |
| $D_3$    | $E \rightarrow A_2$              | $D_6$    | $E_1, E_2 \rightarrow A_2$                          |
| $C_{3v}$ | $E \rightarrow A_2$              | $C_{6v}$ | $E_1, E_2 \rightarrow A_2$                          |
| $D_{3d}$ | $E_g, E_u \rightarrow A_{2g}$    | $D_{6h}$ | $E_{1g}, E_{2g}, E_{1u}, E_{2u} \rightarrow A_{2g}$ |
| $C_{3h}$ | $E', E'' \rightarrow A'$         | $T$      | $E, T \rightarrow A$                                |
| $D_{3h}$ | $E', E'' \rightarrow A'_2$       | $T_h$    | $E_g, E_u, T_g \rightarrow A_g$                     |
| $C_4$    | $E \rightarrow A$                |          | $T_u \rightarrow A_u$                               |
| $S_4$    | $E \rightarrow A$                | $O$      | $E, T_2 \rightarrow A_2$                            |
| $C_{4h}$ | $E_g, E_u \rightarrow A_g$       |          | $T_1 \rightarrow A_1$                               |
| $D_4$    | $E \rightarrow A_2$              | $T_d$    | $E, T_2 \rightarrow A_2$                            |
| $C_{4v}$ | $E \rightarrow A_2$              |          | $T_1 \rightarrow A_1$                               |
| $D_{2d}$ | $E \rightarrow A_2$              | $O_h$    | $E_g, E_u, T_{2g} \rightarrow A_{2g}$               |
| $D_{4h}$ | $E_g, E_u \rightarrow A_{2g}$    |          | $T_{1g} \rightarrow A_{1g}$                         |
|          |                                  |          | $T_{1u} \rightarrow A_{1u}$                         |
|          |                                  |          | $T_{2u} \rightarrow A_{2u}$                         |

TABLE II. Determinant representations for real irreducible representations of dimension two or three in the crystallographic point groups  $P$ . Point groups with only one-dimensional real irreducible representations are omitted.

zero (Fig. 1), and identical conditions apply to the non-trivial powers of one- and two-dimensional representations.

Reality closes the wave-vector content of  $\Gamma$  under  $\mathbf{k} \rightarrow -\mathbf{k}$ , so any unpaired entry must be a time-reversal invariant momentum (TRIM)  $\mathbf{K}$  with  $2\mathbf{K} \equiv 0$ . A three-dimensional  $\Gamma$  thus has content  $\{\mathbf{k}, -\mathbf{k}, \mathbf{K}\}$  or  $\{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3\}$ , and a two-dimensional one  $\{\mathbf{k}, -\mathbf{k}\}$  or  $\{\mathbf{K}_1, \mathbf{K}_2\}$ . For the content  $\{\mathbf{k}, -\mathbf{k}, \mathbf{K}\}$ ,  $\Lambda^2\Gamma$  always reaches  $\mathbf{k} = 0$  and  $\Lambda^3\Gamma$  does so only for  $\mathbf{K} \equiv 0$ ; for three TRIMs,  $\Lambda^2\Gamma$  requires two of them to coincide, since  $2\mathbf{K} \equiv 0$  turns  $\mathbf{K}_j + \mathbf{K}_k \equiv 0$  into  $\mathbf{K}_j \equiv \mathbf{K}_k$ , and  $\Lambda^3\Gamma$  requires them to sum to zero.

#### IV. OBSERVABLES

We now apply this representation hierarchy to several observables, expressing the symmetry constraints directly in terms of the representation  $\Gamma$ . For each observable the spin transformation selects one of  $\Lambda^1\Gamma$ ,  $\Lambda^2\Gamma$ , or  $\Lambda^3\Gamma$ , while its spatial tensor character determines the representation with which that exterior power must overlap. Here two real representations are said to overlap when they share a real irreducible component, or equivalently when their tensor product contains the trivial irrep. [35]

Throughout we use  $\Gamma_\ell^\pm$  to denote the rank- $\ell$  irreducible representation of  $O(3)$  with inversion parity  $\pm$ , regarded as a  $\mathbf{k} = 0$  representation of  $G$ , or equivalently of its point group  $P \equiv G/T$ , where  $T$  is the translation subgroup of  $G$ . Thus  $\Gamma_0^+$  is the trivial irrep,  $\Gamma_1^-$  and  $\Gamma_1^+$  are the polar- and axial-vector representations, and  $\Gamma_2^+$  is the quadrupole representation. Note that this superscript is an inversion parity, whereas on  $\Sigma^\pm$ ,  $\alpha^\pm$  and  $\sigma^\pm$  below it denotes time-reversal parity.

#### A. Magnetization

We start by formulating a basic question in this representation-first picture: which of these spin space groups allow a net spin magnetization? Consider the spin space group corresponding to the representation  $\Gamma$ . The net (spin) magnetization  $\mathbf{M}$  [36] is a uniform spin vector, so the spatial part of an operation leaves it unchanged and it transforms under  $\Lambda^1\Gamma = \Gamma$ . For a non-coplanar order,

$$\mathbf{M} \xrightarrow{[\Gamma(g)\|\mathbf{g}]} \Gamma(g)\mathbf{M}, \quad (14)$$

where  $g \in G$ . A magnetization allowed in this phase must then be invariant,

$$\mathbf{M} = \Gamma(g)\mathbf{M}, \quad (15)$$

for all  $g \in G$ , and so spans a one-dimensional subspace left invariant by the group; that is,  $\Gamma$  contains a copy of the trivial irrep.

For collinear or coplanar orders the same condition applies to the one- or two-dimensional representations. The spin-only group in the collinear case further restricts  $\mathbf{M} = M\hat{\mathbf{N}}$  and in the coplanar case to  $\mathbf{M} = M_1\hat{\mathbf{N}}_1 + M_2\hat{\mathbf{N}}_2$  where  $\hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2$  define the preferred plane. Together, we can say:

**Net Spin Magnetization:** A spin space group defined by a real representation  $\Gamma$  allows a net spin magnetization if and only if  $\Gamma$  contains the trivial irrep  $\Gamma_0^+$ .

We call a state *compensated* when  $\Gamma$  does not contain  $\Gamma_0^+$ , so that its net moment vanishes by symmetry. If all parts of the representation  $\Gamma$  are at finite wave vector it is automatically compensated, as in a conventional antiferromagnet, since the trivial irrep is at  $\mathbf{k} = 0$ . The same argument applies to every observable below that is controlled by  $\Lambda^1\Gamma$ , since each must overlap a representation at  $\mathbf{k} = 0$ : for such an order piezomagnetism, the time-odd spin conductivity  $\Sigma^-$ , the time-odd Edelstein response  $\alpha^-$  and the even-rank spin texture all vanish as well (Sections IV B to IV E).

The magnetization may still vanish despite being allowed by symmetry. An example of recent interest is the class of so-called ‘‘Luttinger-compensated’’ magnets [37, 38], collinear insulators or half-metals in which at least one spin channel is gapped. Due to the constraints from Luttinger’s theorem [39] (or, nonperturbatively, Oshikawa’s argument [40, 41]), the net magnetization can only take on quantized values, and thus may exactly vanish even when the symmetry allows a finite magnetization.

Potential examples are the double perovskite  $\text{LaMn}_2\text{SbO}_6$ , identified as a compensated magnetic semiconductor [42, 43], the  $A$  phase of the semiconducting calcium ferrite  $\text{CaFe}_2\text{O}_4$  [44–46], and the polar insulator  $\text{Mn}_2\text{Mo}_3\text{O}_8$  [47–49]. In each,  $\Gamma$  is the trivial irrep ( $A_g$  of  $P4_2/n$  for  $\text{LaMn}_2\text{SbO}_6$ ,  $A_g$  of  $Pnma$  for  $\text{CaFe}_2\text{O}_4$  and  $A_1$  of  $P6_3mc$  for  $\text{Mn}_2\text{Mo}_3\text{O}_8$ ), so a net moment is allowed by symmetry, yet the experimentally refined structures have zero net magnetization, the two inequivalent sublattices of  $\text{Mn}^{2+}$  or  $\text{Fe}^{3+}$  carrying equal and

opposite moments [24]. [50] For  $\text{Mn}_2\text{Mo}_3\text{O}_8$  this zero is a ground-state property: it is an L-type ferrimagnet,[51] whose two sublattices are unequally polarized at finite temperature, giving a net moment below  $T_C \approx 41.5$  K that vanishes only as  $T \rightarrow 0$  [49]. We will encounter a similar non-symmetry-based constraint in Section IV E for the Edelstein effect in collinear magnets.

### B. Piezomagnetism

Next we consider piezomagnetism, defined as the generation of a net magnetic moment in the presence of a finite strain [13, 15–17]. Explicitly,

$$\mathbf{M} = \sum_{\mu\nu} \mathbf{C}_{\mu\nu} \epsilon_{\mu\nu} + \dots \quad (16)$$

For this to be allowed in a spin space group the coefficient  $\mathbf{C}_{\mu\nu}$  must transform trivially. Like the magnetization,  $\mathbf{C}_{\mu\nu}$  carries a single spin index and so transforms under  $\Lambda^1 \Gamma = \Gamma$  in spin space. We assume that  $\Gamma$  does not contain any copies of the trivial irrep, so that the state has no intrinsic magnetization in the absence of strain.

The symmetric strain tensor,  $\epsilon$ , transforms as the symmetric square of the polar-vector representation  $\Gamma_1^-$  of the point group  $P$ . We denote this symmetric square by  $\text{sym}^2(\Gamma_1^-)$ . For non-coplanar order, under a transformation  $[\Gamma(g)||g]$  with  $g \equiv [R|t]$  we have

$$\mathbf{M} \rightarrow \Gamma(g) \mathbf{M}, \quad (17a)$$

$$\epsilon \rightarrow \mathbf{R} \epsilon \mathbf{R}^\top. \quad (17b)$$

Matching these two requires that  $\text{sym}^2(\Gamma_1^-) \otimes \Gamma$  contains the trivial irrep. We can write the symmetric square in a different way as

$$\text{sym}^2(\Gamma_1^-) = \Gamma_0^+ \oplus \Gamma_2^+, \quad (18)$$

where  $\Gamma_0^+$  is the trivial irrep and  $\Gamma_2^+$  is the quadrupole representation. Thus piezomagnetism is allowed by symmetry in a compensated spin space group when  $\Gamma$  overlaps with  $\Gamma_2^+$ . Without compensation the trivial part of the strain, a uniform dilation, can also change the magnetization, and the condition becomes an overlap with  $\Gamma_0^+ \oplus \Gamma_2^+$ . An identical argument applies to collinear and coplanar orders, with the spin-only groups only restricting the allowed direction of  $\mathbf{C}_{\mu\nu}$  in spin space.

**Piezomagnetism:** A compensated spin space group defined by a real representation  $\Gamma$  allows (linear) piezomagnetism if and only if  $\Gamma$  overlaps with the quadrupole representation  $\Gamma_2^+$ .

We give a listing of the polar-vector, axial-vector and quadrupole representations  $\Gamma_1^-$ ,  $\Gamma_1^+$  and  $\Gamma_2^+$  for each of the crystallographic point groups in Table III.

| $P$      | $\Gamma_1^-$                         | $\Gamma_1^+$                         | $\Gamma_2^+$                                     |
|----------|--------------------------------------|--------------------------------------|--------------------------------------------------|
| $C_1$    | $3A$                                 | $3A$                                 | $5A$                                             |
| $C_i$    | $3A_u$                               | $3A_g$                               | $5A_g$                                           |
| $C_2$    | $A \oplus 2B$                        | $A \oplus 2B$                        | $3A \oplus 2B$                                   |
| $C_s$    | $2A' \oplus A''$                     | $A' \oplus 2A''$                     | $3A' \oplus 2A''$                                |
| $C_{2h}$ | $A_u \oplus 2B_u$                    | $A_g \oplus 2B_g$                    | $3A_g \oplus 2B_g$                               |
| $D_2$    | $B_1 \oplus B_2 \oplus B_3$          | $B_1 \oplus B_2 \oplus B_3$          | $2A \oplus B_1 \oplus B_2 \oplus B_3$            |
| $C_{2v}$ | $A_1 \oplus B_1 \oplus B_2$          | $A_2 \oplus B_1 \oplus B_2$          | $2A_1 \oplus A_2 \oplus B_1 \oplus B_2$          |
| $D_{2h}$ | $B_{1u} \oplus B_{2u} \oplus B_{3u}$ | $B_{1g} \oplus B_{2g} \oplus B_{3g}$ | $2A_g \oplus B_{1g} \oplus B_{2g} \oplus B_{3g}$ |
| $C_4$    | $A \oplus E$                         | $A \oplus E$                         | $A \oplus 2B \oplus E$                           |
| $S_4$    | $B \oplus E$                         | $A \oplus E$                         | $A \oplus 2B \oplus E$                           |
| $C_{4h}$ | $A_u \oplus E_u$                     | $A_g \oplus E_g$                     | $A_g \oplus 2B_g \oplus E_g$                     |
| $D_4$    | $A_2 \oplus E$                       | $A_2 \oplus E$                       | $A_1 \oplus B_1 \oplus B_2 \oplus E$             |
| $C_{4v}$ | $A_1 \oplus E$                       | $A_2 \oplus E$                       | $A_1 \oplus B_1 \oplus B_2 \oplus E$             |
| $D_{2d}$ | $B_2 \oplus E$                       | $A_2 \oplus E$                       | $A_1 \oplus B_1 \oplus B_2 \oplus E$             |
| $D_{4h}$ | $A_{2u} \oplus E_u$                  | $A_{2g} \oplus E_g$                  | $A_{1g} \oplus B_{1g} \oplus B_{2g} \oplus E_g$  |
| $C_3$    | $A \oplus E$                         | $A \oplus E$                         | $A \oplus 2E$                                    |
| $S_6$    | $A_u \oplus E_u$                     | $A_g \oplus E_g$                     | $A_g \oplus 2E_g$                                |
| $D_3$    | $A_2 \oplus E$                       | $A_2 \oplus E$                       | $A_1 \oplus 2E$                                  |
| $C_{3v}$ | $A_1 \oplus E$                       | $A_2 \oplus E$                       | $A_1 \oplus 2E$                                  |
| $D_{3d}$ | $A_{2u} \oplus E_u$                  | $A_{2g} \oplus E_g$                  | $A_{1g} \oplus 2E_g$                             |
| $C_6$    | $A \oplus E_1$                       | $A \oplus E_1$                       | $A \oplus E_1 \oplus E_2$                        |
| $C_{3h}$ | $A'' \oplus E'$                      | $A' \oplus E''$                      | $A' \oplus E' \oplus E''$                        |
| $C_{6h}$ | $A_u \oplus E_{1u}$                  | $A_g \oplus E_{1g}$                  | $A_g \oplus E_{1g} \oplus E_{2g}$                |
| $D_6$    | $A_2 \oplus E_1$                     | $A_2 \oplus E_1$                     | $A_1 \oplus E_1 \oplus E_2$                      |
| $C_{6v}$ | $A_1 \oplus E_1$                     | $A_2 \oplus E_1$                     | $A_1 \oplus E_1 \oplus E_2$                      |
| $D_{3h}$ | $A_2' \oplus E'$                     | $A_2' \oplus E''$                    | $A_1' \oplus E' \oplus E''$                      |
| $D_{6h}$ | $A_{2u} \oplus E_{1u}$               | $A_{2g} \oplus E_{1g}$               | $A_{1g} \oplus E_{1g} \oplus E_{2g}$             |
| $T$      | $T$                                  | $T$                                  | $E \oplus T$                                     |
| $T_h$    | $T_u$                                | $T_g$                                | $E_g \oplus T_g$                                 |
| $O$      | $T_1$                                | $T_1$                                | $E \oplus T_2$                                   |
| $T_d$    | $T_2$                                | $T_1$                                | $E \oplus T_2$                                   |
| $O_h$    | $T_{1u}$                             | $T_{1g}$                             | $E_g \oplus T_{2g}$                              |

TABLE III. Polar-vector representation  $\Gamma_1^-$ , axial-vector representation  $\Gamma_1^+$  and quadrupole representation  $\Gamma_2^+$  for each crystallographic point group  $P$ . The quadrupole representation  $\Gamma_2^+$  is the traceless part of the symmetric square  $\text{sym}^2(\Gamma_1^-) = \Gamma_0^+ \oplus \Gamma_2^+$ , the trivial irrep  $\Gamma_0^+$  carrying the trace. Note that  $\Gamma_2^+$  may itself contain copies of  $\Gamma_0^+$  in some lower-symmetry groups.

### C. Spin and Charge Transport

We next consider constraints on charge and spin transport. The symmetry-imposed form of transport tensors has been determined for magnetic point groups [19, 52], while spin transport in the non-relativistic limit has been discussed in specific magnetic systems [53, 54] and more generally using spin-group symmetry [12, 13]. Define a charge current  $j_\nu$  and a spin current  $J_\nu^\mu$ , [55] where  $\mu$  is a spin index and  $\nu$  is a spatial index. The spin and charge currents are related to the electric field  $E_\nu$  and the field gradient  $\partial_\nu B^\mu$  by conductivity tensors

$$j_\nu = \sum_\rho \sigma_{\nu\rho} E_\rho + \sum_{\mu\rho} \tilde{\Sigma}_{\nu\rho}^\mu \partial_\rho B^\mu, \quad (19a)$$

$$J_\nu^\mu = \sum_\rho \Sigma_{\nu\rho}^\mu E_\rho + \sum_{\mu'\rho} S_{\nu\rho}^{\mu\mu'} \partial_\rho B^{\mu'}. \quad (19b)$$

Equations (19a) and (19b) contain the usual charge conductivity  $\sigma_{\nu\rho}$ , the spin conductivities  $\Sigma_{\nu\rho}^\mu$  and  $\tilde{\Sigma}_{\nu\rho}^\mu$ , and the spin-spin

conductivity  $S_{\nu\rho}^{\mu\mu'}$ , which we do not consider further. Spin currents of this kind underlie the forward and inverse spin Hall effects [56–59], and can be generated in compensated antiferromagnets even in the absence of spin-orbit coupling [18].

### 1. Spin Conductivity

The spin conductivity  $\Sigma_{\nu\rho}^{\mu}$  describes the generation of a spin current in response to an electric field. Its partner  $\tilde{\Sigma}_{\nu\rho}^{\mu}$  describes the reciprocal response of a charge current to a field conjugate to the spin density. Onsager–Casimir reciprocity [60, 61] gives

$$\Sigma_{\nu\rho}^{\mu}(H) = -\tilde{\Sigma}_{\rho\nu}^{\mu}(THT^{-1}), \quad (20)$$

where  $H$  is the system Hamiltonian and  $T$  is the (antiunitary) time reversal operator. Since  $\Sigma$  and  $\tilde{\Sigma}$  are exchanged by time reversal, neither has a definite time-reversal parity on its own. We therefore define

$$\Sigma_{\nu\rho}^{\mu,\pm} \equiv \frac{1}{2}(\Sigma_{\nu\rho}^{\mu} \mp \tilde{\Sigma}_{\rho\nu}^{\mu}), \quad (21)$$

where  $\Sigma^-$  is time odd and  $\Sigma^+$  is time even.

These combinations have definite transformation laws under the spin-space symmetries. Consider a non-coplanar spin space group defined by a real three-dimensional representation  $\Gamma$  of the parent space group  $G$ . Taking into account the sign of  $\det \Gamma(g)$ , which indicates time reversal, we have

$$\Sigma_{\nu\rho}^- \xrightarrow{[\Gamma(g)\|g]} \sum_{\nu'\rho'} \Gamma(g) \Sigma_{\nu'\rho'}^- R_{\rho\rho'}(g) R_{\nu\nu'}(g), \quad (22a)$$

$$\Sigma_{\nu\rho}^+ \xrightarrow{[\Gamma(g)\|g]} \det \Gamma(g) \sum_{\nu'\rho'} \Gamma(g) \Sigma_{\nu'\rho'}^+ R_{\rho\rho'}(g) R_{\nu\nu'}(g), \quad (22b)$$

where  $\mathbf{R}(g)$  is the rotational part of the space group operation  $g = [\mathbf{R}|\mathbf{t}]$ . The spin conductivity therefore involves  $\Gamma$  for the time-odd component and the determinant twist  $\Gamma \otimes \det \Gamma = \Lambda^2 \Gamma$  of Eq. (7a) for the time-even component, while both spatial indices transform as ordinary vectors. The spin index of  $\Sigma^+$  is thus a spin pseudovector, like the cross product in Eq. (8): it is even under spin inversion, as a time-even quantity must be, while that of  $\Sigma^-$  is odd.

Explicitly, the time-odd spin conductivity transforms as

$$\Gamma \otimes (\Gamma_1^- \otimes \Gamma_1^-) = \Gamma \otimes (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+). \quad (23)$$

Here  $\Gamma_0^+ \oplus \Gamma_2^+ = \text{sym}^2(\Gamma_1^-)$  is the part symmetric in the spatial indices [Eq. (18)] and  $\Gamma_1^+ = \Lambda^2 \Gamma_1^-$  the antisymmetric part. We thus require that  $\Gamma$  have nonzero overlap with  $\Gamma_0^+$ ,  $\Gamma_1^+$  or  $\Gamma_2^+$ . The time-even part follows from the same argument, with the spin index now transforming as  $\Lambda^2 \Gamma$  rather than in  $\Gamma$  itself, giving

$$\Lambda^2 \Gamma \otimes (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+). \quad (24)$$

Therefore the time-even spin conductivity is allowed if  $\Lambda^2 \Gamma$  has nonzero overlap with  $\Gamma_0^+$ ,  $\Gamma_1^+$  or  $\Gamma_2^+$ .

The spin-only groups impose additional constraints on collinear and coplanar orders. For a collinear magnet, the residual spin rotation symmetry about the ordering axis  $\hat{N}$  requires

$$\Sigma_{\nu\rho}^{\mu} = \Sigma_{\nu\rho} \hat{N}^{\mu}. \quad (25)$$

Time reversal combined with a two-fold spin rotation perpendicular to  $\hat{N}$  then forces the time-even component to vanish,

$$\Sigma_{\nu\rho}^{\mu,+} = 0. \quad (26)$$

We thus require the one-dimensional irrep  $\Gamma$  defining the collinear order to have nonzero overlap with  $\Gamma_0^+$ ,  $\Gamma_1^+$  or  $\Gamma_2^+$  for the time-odd part to be allowed.

For coplanar order, let  $\hat{N}_1$  and  $\hat{N}_2$  span the preferred plane. The spin-only symmetry instead requires

$$\Sigma_{\nu\rho}^- = \Sigma_{1,\nu\rho} \hat{N}_1 + \Sigma_{2,\nu\rho} \hat{N}_2, \quad (27a)$$

$$\Sigma_{\nu\rho}^+ = \Sigma_{3,\nu\rho} (\hat{N}_1 \times \hat{N}_2). \quad (27b)$$

Thus the time-odd response lies in the ordering plane while the time-even response is perpendicular to it. In terms of the two-dimensional representation  $\Gamma$ , the in-plane parts require  $\Gamma$  to overlap with  $\Gamma_0^+$ ,  $\Gamma_1^+$  or  $\Gamma_2^+$ , while the out-of-plane part requires overlap of  $\det \Gamma = \Lambda^2 \Gamma$  with the same.

These conditions for collinear, coplanar and non-coplanar orders can be encapsulated in the following unified statements:

**Time-Odd Spin Conductivity:** A spin space group defined by a real representation  $\Gamma$  allows time-odd spin conductivity if and only if  $\Gamma$  has nonzero overlap with  $\Gamma_0^+$ ,  $\Gamma_1^+$  or  $\Gamma_2^+$ .

**Time-Even Spin Conductivity:** A spin space group defined by a real representation  $\Gamma$  allows time-even spin conductivity if and only if  $\Lambda^2 \Gamma$  has nonzero overlap with  $\Gamma_0^+$ ,  $\Gamma_1^+$ , or  $\Gamma_2^+$ .

Note that for a one-dimensional representation  $\Lambda^2 \Gamma = 0$ , so this also encapsulates that the time-even response vanishes for collinear order.

Because  $\Gamma_0^+$ ,  $\Gamma_1^+$  and  $\Gamma_2^+$  all live at  $\mathbf{k} = 0$ , they also constrain which wave vectors may appear in  $\Gamma$ , and these restrictions follow from Section III. The time-odd response is controlled by  $\Gamma \equiv \Lambda^1 \Gamma$  and therefore requires a  $\mathbf{k} = 0$  component to be present in  $\Gamma$ , while the time-even response is controlled by  $\Lambda^2 \Gamma$  and requires only a pair of wave vectors that sum to zero. For coplanar order this is a pair  $\mathbf{k}, -\mathbf{k}$ ; for non-coplanar order  $\Gamma$  must contain such a pair together with a TRIM  $\mathbf{K}$ .

For collinear order the conventional band conductivity can obey a further restriction that does not follow from the transformation laws alone. The spin-only symmetry combining time reversal with a two-fold spin rotation perpendicular to  $\hat{N}$  acts within each spin sector as spinless time reversal. If the two sectors conduct independently, as for non-interacting electrons,

$\Sigma_{\nu\rho} \propto \sigma_{\nu\rho}^{\uparrow} - \sigma_{\nu\rho}^{\downarrow}$  [Eq. (25)] is a difference of spin-resolved charge conductivities that are each symmetric by Onsager reciprocity under this spinless time reversal. Then  $\Sigma^-$  is symmetric in its spatial indices and thus requires  $\Gamma \cap (\Gamma_0^+ \oplus \Gamma_2^+) \neq 0$ ; for compensated order this is identical to the piezomagnetic condition. This symmetric response is the spin-splitter effect of González-Hernández *et al.* [19].

However, an antisymmetric part is not excluded once the sectors are coupled, for example by spin Coulomb drag [62], which violates the assumption of independent spin channels. The purely antisymmetric channel is the only one allowed, for example, in the reported 300 K structure of  $\alpha$ -Fe<sub>2</sub>O<sub>3</sub> [24, 63], above the Morin transition, whose  $\Gamma = A_{2g}$  of  $R\bar{3}c$  overlaps with  $\Gamma_1^+ = A_{2g} \oplus E_g$  but not with  $\Gamma_0^+$  nor  $\Gamma_2^+ = A_{1g} \oplus 2E_g$ , so that piezomagnetism and the symmetric part of  $\Sigma^-$  are also forbidden. [64] The same holds for the chalcopyrite CuFeS<sub>2</sub> [24, 65], whose  $\Gamma = A_2$  of  $I\bar{4}2d$  likewise overlaps  $\Gamma_1^+ = A_2 \oplus E$  but not  $\Gamma_0^+$  nor  $\Gamma_2^+$ . Any antisymmetric spin conductivity arising in these materials therefore requires interactions that couple the spin channels.

## 2. Anomalous Hall Conductivity

The charge conductivity follows from the same construction with the spin index removed, so that only the determinant of  $\Gamma(g)$  enters. Define the symmetric and antisymmetric parts of the charge conductivity as

$$\sigma_{\nu\rho}^{\pm} \equiv \frac{1}{2}(\sigma_{\nu\rho} \pm \sigma_{\rho\nu}). \quad (28)$$

The symmetric part is time-even while the antisymmetric part is time-odd, so the superscripts again denote time-reversal parity. For a non-coplanar order a spin-space operation  $[\Gamma(g)||g]$  acts as

$$\sigma_{\nu\rho}^+ \xrightarrow{[\Gamma(g)||g]} \sum_{\nu'\rho'} \sigma_{\nu'\rho'}^+ R_{\rho\rho'}(g) R_{\nu\nu'}(g), \quad (29a)$$

$$\sigma_{\nu\rho}^- \xrightarrow{[\Gamma(g)||g]} \det \Gamma(g) \sum_{\nu'\rho'} \sigma_{\nu'\rho'}^- R_{\rho\rho'}(g) R_{\nu\nu'}(g). \quad (29b)$$

In representation theory language this states that

$$\sigma^+ \sim \Gamma_0^+ \oplus \Gamma_2^+, \quad (30a)$$

$$\sigma^- \sim \det \Gamma \otimes \Gamma_1^+. \quad (30b)$$

The symmetric conductivity is therefore always allowed. The antisymmetric part is equivalent to a Hall pseudovector, an axial vector twisted by the determinant representation. That representation is the top exterior power  $\Lambda^3 \Gamma$ , and can be read directly from Table II when  $\Gamma$  is irreducible.

Collinear and coplanar states possess spin-only symmetries that force  $\sigma^- = 0$ , while a non-coplanar state can exhibit a finite anomalous Hall conductivity even without relativistic interactions [66–68].

**Anomalous Hall Conductivity:** A spin space group defined by a real representation  $\Gamma$  allows anomalous Hall conductivity in the non-relativistic limit if and only if  $\Lambda^3 \Gamma$  has nonzero overlap with the axial-vector representation  $\Gamma_1^+$ .

Since  $\Lambda^3 \Gamma = 0$  for collinear and coplanar orders this encodes that they cannot support an anomalous Hall effect. The same symmetry analysis applies unchanged to the thermal conductivity and the thermal Hall conductivity.

The Hall pseudovector transforms exactly as the orbital magnetization, an axial spatial vector odd under time reversal, rather than as the spin magnetization, which transforms under  $\Gamma$ , so in the non-relativistic limit a Hall response neither requires nor implies a net spin magnetization. This non-relativistic Hall response was termed the *geometrical Hall effect* in Watanabe *et al.* [12]; related discussions are given in Refs. [13, 69]. Its determinant factor matches that of the scalar spin chirality, Eq. (9), whose microscopic counterpart is the Berry phase accumulated by an electron hopping through a chiral spin background [66, 70, 71]. This correspondence is purely a statement about symmetry. It does not require a particular microscopic chirality to be nonzero, nor does it identify the mechanism responsible for an allowed Hall response.

Since the Hall response is controlled by  $\Lambda^3 \Gamma$ , its translation selection rule is that the three wave vectors sum to zero, so  $\Gamma$  need not contain a  $\mathbf{k} = 0$  component: its content is either  $\mathbf{0}$ ,  $\mathbf{k}$ ,  $-\mathbf{k}$  with  $\mathbf{k}$  arbitrary, or three TRIM points  $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$  with  $2\mathbf{K}_n \equiv \mathbf{0}$  and  $\mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3 \equiv \mathbf{0}$ .

Two general restrictions on the non-relativistic anomalous Hall response follow from the fact that  $\Lambda^3 \Gamma = \det \Gamma$  is one-dimensional.

First, for the cubic point groups  $T, T_h, O, T_d$  and  $O_h$  the axial-vector representation  $\Gamma_1^+$  is a single three-dimensional irrep, so the overlap vanishes for every  $\Gamma$  and a cubic parent point group forbids the anomalous Hall effect outright, as for pyrochlore FeF<sub>3</sub> in Section V D.

Second, at  $\mathbf{k} = 0$  the representation  $\Gamma$  is one of the point group, and outside the cubic groups it is either a sum of three one-dimensional irreps or a one-dimensional irrep together with a two-dimensional one. Writing  ${}^d\Gamma$  for a  $d$ -dimensional irreducible representation, the second case has  $\det \Gamma = {}^1\Gamma \otimes \det({}^2\Gamma)$  (Appendix A), and comparing Tables II and III shows that  $\det({}^2\Gamma)$  is the one-dimensional part of  $\Gamma_1^+$  in every such group. Overlap then requires  ${}^1\Gamma = \Gamma_0^+$  and so the order is uncompensated. A compensated anomalous Hall response from  $\mathbf{k} = 0$  order therefore requires  $\Gamma$  to be a sum of three one-dimensional irreps; orders built from layers or sublattices that are each collinear, with moments along three non-coplanar axes, may be a natural place to realize this. The reported structure of the polar magnet DyCrWO<sub>6</sub> [24, 72] illustrates a version of this case: its compensated non-coplanar  $\mathbf{k} = 0$  order, combining a noncollinear Dy<sup>3+</sup> sublattice with a collinear Cr<sup>3+</sup> one, has  $\Gamma = A_2 \oplus 2B_1$  of  $Pna2_1$ , so that  $\det \Gamma = A_2$  overlaps  $\Gamma_1^+ = A_2 \oplus B_1 \oplus B_2$ . Since the Dy<sup>3+</sup> moments likely make the magnetism of this compound relativistic, we use its structure only as an illustration.

| Response                                                                                               | Required overlap                                                         | Wave-vectors in $\Gamma$                                                                                                           |
|--------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|
| Collinear order                                                                                        |                                                                          |                                                                                                                                    |
| $\Sigma^-$                                                                                             | $\Lambda^1 \Gamma \cap (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+)$ | $\mathbf{0}$                                                                                                                       |
| $\Sigma^+$                                                                                             | —                                                                        | —                                                                                                                                  |
| $\sigma^-$                                                                                             | —                                                                        | —                                                                                                                                  |
| Coplanar order, $\Lambda^2 \Gamma = \det \Gamma$                                                       |                                                                          |                                                                                                                                    |
| $\Sigma^-$                                                                                             | $\Lambda^1 \Gamma \cap (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+)$ | $\mathbf{0}$                                                                                                                       |
| $\Sigma^+$                                                                                             | $\Lambda^2 \Gamma \cap (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+)$ | $\mathbf{k}, -\mathbf{k}$                                                                                                          |
| $\sigma^-$                                                                                             | —                                                                        | —                                                                                                                                  |
| Non-coplanar order, $\Lambda^2 \Gamma = \Gamma \otimes \det \Gamma$ , $\Lambda^3 \Gamma = \det \Gamma$ |                                                                          |                                                                                                                                    |
| $\Sigma^-$                                                                                             | $\Lambda^1 \Gamma \cap (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+)$ | $\mathbf{0}$                                                                                                                       |
| $\Sigma^+$                                                                                             | $\Lambda^2 \Gamma \cap (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+)$ | $\mathbf{k}, -\mathbf{k}, \mathbf{K}$                                                                                              |
| $\sigma^-$                                                                                             | $\Lambda^3 \Gamma \cap \Gamma_1^+$                                       | $\mathbf{0}, \mathbf{k}, -\mathbf{k}$ or<br>$\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ ( $\sum_n \mathbf{K}_n \equiv \mathbf{0}$ ) |

TABLE IV. Conditions for the time-odd and time-even spin conductivities and the anomalous Hall conductivity, by type of magnetic order, where  $\cap$  denotes a nonzero overlap of representations and a dash a response forbidden by spin-only symmetry. The last column gives the wave vectors the representation must contain:  $\mathbf{k}$  is arbitrary, while  $\mathbf{K}$  always denotes a time-reversal invariant momentum,  $2\mathbf{K} \equiv \mathbf{0}$ . The wave-vector conditions are derived in Section III.

These conditions for the spin and charge transport tensors, in terms of the representations and the wave-vector content are summarized in Table IV.

#### D. Momentum-Space Spin Textures

The same representation-first viewpoint can be applied to the momentum-space spin texture of the electronic bands [73]. Let  $s(\mathbf{q})$  denote the energy-resolved spin spectral density at a fixed energy  $\epsilon$ , where  $\mathbf{q}$  is the crystal momentum, distinguished here from the ordering wave vector  $\mathbf{k}$ . This quantity is a distribution supported on the constant-energy surfaces  $\epsilon = \epsilon_n(\mathbf{q})$ . Explicitly, we define

$$s(\mathbf{q}) \equiv \text{tr} [s\delta(\epsilon - H(\mathbf{q}))] \quad (31)$$

where  $H(\mathbf{q})$  is the Bloch Hamiltonian at wave vector  $\mathbf{q}$  and  $s$  the electronic spin operator. The trace over electronic states makes this definition independent of the choice of basis within any degenerate subspaces.

Under a spin-space operation  $[\Gamma(g)||g]$  with  $g \equiv [\mathbf{R}|\mathbf{t}]$  the spin is rotated by  $\Gamma(g)$  while the momentum is rotated by  $\mathbf{R}(g)$  and, when the operation includes time reversal, reversed. For a non-coplanar order the momentum-space spin texture is thus transformed as

$$s(\mathbf{q}) \xrightarrow{[\Gamma(g)||g]} \Gamma(g)s(\det \Gamma(g)\mathbf{R}^\top(g)\mathbf{q}). \quad (32)$$

The factor of  $\det \Gamma(g)$  is the momentum reversal accompanying time reversal. Invariance within the ordered phase requires the right-hand side to equal  $s(\mathbf{q})$  for every  $g \in G$ .

More concretely, we can characterize the texture  $s(\mathbf{q})$  by its projections onto momentum-space lattice harmonics. Since  $\mathbf{q}$

transforms as the polar-vector representation  $\Gamma_1^-$ , its traceless rank- $\ell$  multipoles transform as  $\Gamma_\ell^{(-1)^\ell}$ . Let  $f_{\ell a}(\mathbf{q})$  be a basis of real periodic functions on the Brillouin zone transforming in this representation, where  $a$  labels its distinct components. The moments

$$\int_{\text{BZ}} d^3q s(\mathbf{q}) f_{\ell a}(\mathbf{q}) \quad (33)$$

then identify the corresponding rank- $\ell$  components of the momentum-space spin texture. Near the zone center, the  $f_{\ell a}(\mathbf{q})$  may be chosen to reduce to the usual homogeneous traceless polynomials of degree  $\ell$ . The familiar  $s$ -,  $p$ -,  $d$ -,  $f$ -, and  $g$ -wave terminology therefore corresponds respectively to the multipole representations  $\Gamma_0^+$ ,  $\Gamma_1^-$ ,  $\Gamma_2^+$ ,  $\Gamma_3^-$ , and  $\Gamma_4^+$  (Fig. 2).

The spin-only symmetries of collinear and coplanar order constrain the parity of  $s(\mathbf{q})$  under  $\mathbf{q} \rightarrow -\mathbf{q}$ , and thus which of these multipoles may appear. We write  $s_i(\mathbf{q}) \equiv \hat{N}_i \cdot s(\mathbf{q})$  for its components, with  $\hat{N}_1 \equiv \hat{N}$  for collinear order and  $\hat{N}_3 \equiv \hat{N}_1 \times \hat{N}_2$  for coplanar order. For collinear order, the continuous rotations about  $\hat{N}$  require

$$s(\mathbf{q}) = s_1(\mathbf{q})\hat{N}_1, \quad (34)$$

while the two-fold spin rotation perpendicular to  $\hat{N}$  combined with time reversal further requires

$$s_1(\mathbf{q}) = +s_1(-\mathbf{q}). \quad (35)$$

The spin texture therefore contains only even momentum multipoles, and a contribution of even rank  $\ell$  is allowed when  $\Gamma$  overlaps with  $\Gamma_\ell^+$ . For a compensated collinear order the scalar contribution is absent, so the lowest possible term is the quadrupolar ( $d$ -wave) contribution, present if  $\Gamma$  overlaps with  $\Gamma_2^+$  [Fig. 2(a)]. If this overlap also vanishes the leading spin-texture multipole occurs at still higher even rank, as in the  $g$ -wave texture found in MnTe discussed in Section VB.

For coplanar order, the spin-only symmetry, a two-fold rotation  $\mathbf{R}_{\hat{N}_3}(\pi)$  combined with time reversal, instead requires

$$s_1(\mathbf{q}) = +s_1(-\mathbf{q}), \quad (36a)$$

$$s_2(\mathbf{q}) = +s_2(-\mathbf{q}), \quad (36b)$$

$$s_3(\mathbf{q}) = -s_3(-\mathbf{q}). \quad (36c)$$

The in-plane spin texture is thus even in momentum and transforms under  $\Gamma$  [Fig. 2(b)], while the component along  $\hat{N}_3$  is odd and transforms under  $\det \Gamma$  [Fig. 2(d)]. An even-rank contribution in  $s_1(\mathbf{q})$  or  $s_2(\mathbf{q})$  therefore requires  $\Gamma$  to overlap with  $\Gamma_\ell^+$ , and an odd-rank contribution appears in  $s_3(\mathbf{q})$  if  $\det \Gamma$  overlaps with  $\Gamma_\ell^-$ . In particular, a  $p$ -wave spin texture perpendicular to the ordering plane is possible when  $\det \Gamma$  overlaps with the polar-vector representation  $\Gamma_1^-$ .

For non-coplanar order there is no residual spin-only symmetry fixing the parity of the spin texture, and both even- and odd-rank contributions may occur. The momentum reversal in Eq. (32) contributes a factor of  $\det \Gamma$  for each power of  $\mathbf{q}$ , so an even-rank contribution requires  $\Gamma$  to overlap with  $\Gamma_\ell^+$  [Fig. 2(c)], while an odd-rank contribution requires the determinant twist  $\Gamma \otimes \det \Gamma$  to overlap with  $\Gamma_\ell^-$ . This can be stated

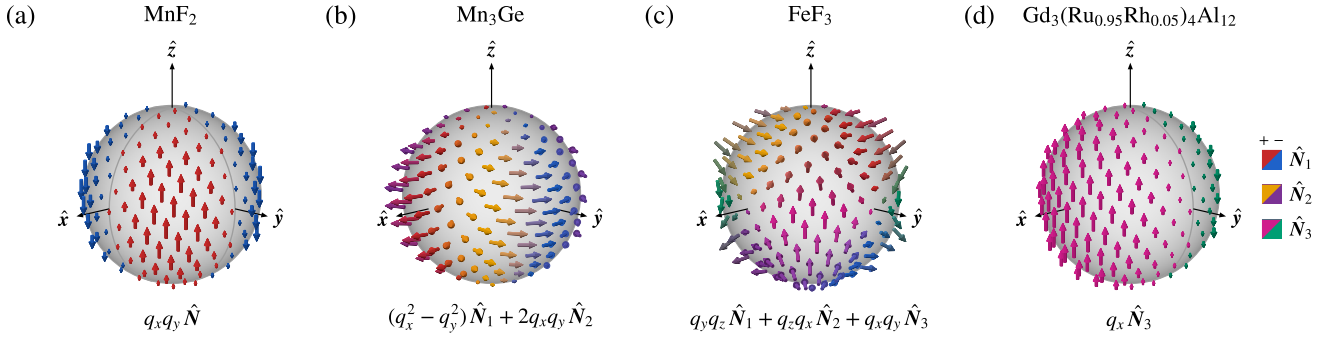


FIG. 2. Momentum-space spin textures  $s(\mathbf{q})$  of four of the examples of Section V, drawn on the sphere of directions  $\hat{\mathbf{q}}$ , with the spin axes laid along the lattice axes. Each arrow is colored by the spin axes it lies along, with a pair of colors for the two signs along each axis, and  $\hat{\mathbf{N}}$  in (a) colored as  $\hat{\mathbf{N}}_1$ . Gray lines mark where the texture vanishes. (a)  $\text{MnF}_2$  (Section V A), a collinear  $d$ -wave texture. (b)  $\text{Mn}_3\text{Ge}$  (Section V C), an in-plane  $d$ -wave texture. (c) Pyrochlore  $\text{FeF}_3$  (Section V D), a three-component  $d$ -wave texture. (d)  $\text{Gd}_3(\text{Ru}_{0.95}\text{Rh}_{0.05})_4\text{Al}_{12}$  (Section V H), a  $p$ -wave texture along  $\hat{\mathbf{N}}_3 = \hat{\mathbf{N}}_1 \times \hat{\mathbf{N}}_2$ , normal to the spin plane and odd in  $\mathbf{q}$ .

more compactly:  $\Gamma \otimes (\det \Gamma)^\ell$  must overlap with  $\Gamma_\ell^{(-1)^\ell}$ . The even- and odd-rank components therefore alternate between  $\Gamma$  and  $\Lambda^2 \Gamma$ , matching the conditions found for collinear and coplanar orders. The common condition for collinear, coplanar and non-coplanar orders can thus be stated compactly as

**Momentum-Space Spin Texture:** A spin space group defined by a real representation  $\Gamma$  allows a rank- $\ell$  contribution to the momentum-space spin texture if and only if  $\Gamma$  overlaps with  $\Gamma_\ell^+$  when  $\ell$  is even, and  $\Lambda^2 \Gamma$  overlaps with  $\Gamma_\ell^-$  when  $\ell$  is odd.

All even-rank contributions require that the representation has components that live at  $\mathbf{k} = 0$ , while the odd-rank contributions may live entirely at finite wave vector. The conditions on the wave vector content of odd-rank parts are identical to those that appear for the time-even spin conductivity (see Section IV C 1). For coplanar order, the odd-rank contributions are only possible when  $\mathbf{k}$  and  $-\mathbf{k}$  are present in  $\Gamma$ . If  $\mathbf{k} \neq 0$  here the even-rank contributions vanish and the state can only support an odd-parity spin texture perpendicular to the ordering plane, as for the odd-parity magnets discussed in Section VI C. For non-coplanar order, the odd-rank contributions are only possible when  $\mathbf{k}$ ,  $-\mathbf{k}$ , and  $\mathbf{K}$  are present in  $\Gamma$  where  $2\mathbf{K} \equiv \mathbf{0}$ , as shown in Section III.

The parity conditions on  $s(\mathbf{q})$ , the required representation overlaps and the possible wave-vector content are collected in Table V for collinear, coplanar and non-coplanar order.

Note that an identical set of symmetry conditions applies to the magnetic moment carried by a magnon excitation, which is closely related to the magnon chirality probed in polarized inelastic neutron scattering. If we define the single magnon state

$$|\mathbf{q}, n\rangle \equiv a_{\mathbf{q}, n}^\dagger |0\rangle, \quad (37)$$

where  $a_{\mathbf{q}, n}^\dagger$  is the creation operator for a magnon in band  $n$  at momentum  $\mathbf{q}$ , then the magnon magnetic moment for that

state is

$$\mathbf{m}_n(\mathbf{q}) \equiv \langle \mathbf{q}, n | \mathbf{M} | \mathbf{q}, n \rangle - \langle 0 | \mathbf{M} | 0 \rangle. \quad (38)$$

The quantity analogous to the spin texture is then energy resolved and summed over all magnon bands

$$\mathbf{m}(\mathbf{q}) \equiv \sum_n \mathbf{m}_n(\mathbf{q}) \delta(\omega - \omega_n(\mathbf{q})) \quad (39)$$

at some fixed energy  $\omega$ , where  $\omega_n(\mathbf{q})$  are the associated magnon energies. This transforms exactly as the spin texture  $s(\mathbf{q})$  does in Eq. (32), so the conditions on its appearance and symmetry are those already derived: its parity under  $\mathbf{q} \rightarrow -\mathbf{q}$  is fixed by the spin-only symmetry of the underlying magnetic order, and the multipoles it may carry are fixed by  $\Gamma$ . [74]

The appearance of chiral magnons is well known in collinear ferromagnets and ferrimagnets, with the opposite polarizations of distinct magnon branches first directly measured in  $\text{Y}_3\text{Fe}_5\text{O}_{12}$  [75] and magnon chirality subsequently measured in the ferromagnet  $\text{SrRuO}_3$  [76], while a momentum-dependent splitting of opposite-chirality branches is a characteristic signature of altermagnets [77, 78]. Magnon chiralities can be accessed with polarized inelastic neutron scattering [79], as has been demonstrated more recently in  $\text{MnF}_2$  [80] and  $\text{MnTe}$  [81].

### E. Edelstein Effect

A closely related consequence of the momentum-space spin texture is the Edelstein effect [23, 82], or current-induced spin polarization. [83] We write its linear response as

$$\mathbf{M}^\mu = \sum_\nu \alpha_\nu^\mu E_\nu. \quad (40)$$

The Edelstein tensor and its Onsager partner, the inverse Edelstein effect [84], may be combined into components that are even and odd under time reversal,  $\alpha^\pm$ , as for the spin conductivity in Section IV C 1. Although the Edelstein response has the same spatial tensor structure as the equilibrium linear magnetoelectric effect [85–88], whose conventional symbol  $\alpha$  we

| Component                                                           | Parity in $\mathbf{q}$ | Required overlap                      | Wave-vectors in $\Gamma$              |
|---------------------------------------------------------------------|------------------------|---------------------------------------|---------------------------------------|
| Collinear order                                                     |                        |                                       |                                       |
| $s_1$                                                               | even                   | $\Lambda^1 \Gamma \cap \Gamma_\ell^+$ | $\mathbf{0}$                          |
| $\mathbf{s} \perp \hat{\mathbf{N}}_1$                               | —                      | —                                     | —                                     |
| Coplanar order, $\Lambda^2 \Gamma = \det \Gamma$                    |                        |                                       |                                       |
| $s_1, s_2$                                                          | even                   | $\Lambda^1 \Gamma \cap \Gamma_\ell^+$ | $\mathbf{0}$                          |
| $s_3$                                                               | odd                    | $\Lambda^2 \Gamma \cap \Gamma_\ell^-$ | $\mathbf{k}, -\mathbf{k}$             |
| Non-coplanar order, $\Lambda^2 \Gamma = \Gamma \otimes \det \Gamma$ |                        |                                       |                                       |
| $s$                                                                 | even                   | $\Lambda^1 \Gamma \cap \Gamma_\ell^+$ | $\mathbf{0}$                          |
|                                                                     | odd                    | $\Lambda^2 \Gamma \cap \Gamma_\ell^-$ | $\mathbf{k}, -\mathbf{k}, \mathbf{K}$ |

TABLE V. Conditions for a rank- $\ell$  contribution to the momentum-space spin texture  $\mathbf{s}(\mathbf{q})$ , by type of magnetic order, with components  $s_i = \hat{\mathbf{N}}_i \cdot \mathbf{s}$  as defined in Section IV D. The parity of a component under  $\mathbf{q} \rightarrow -\mathbf{q}$  fixes which multipoles it may carry: an even component only even  $\ell$ , transforming as  $\Gamma_\ell^+$ , and an odd component only odd  $\ell$ , transforming as  $\Gamma_\ell^-$ . Here  $\cap$  denotes a nonzero overlap of representations and a dash a component that vanishes identically by spin-only symmetry. The last column gives the wave vectors the representation must contain, derived in Section III:  $\mathbf{k}$  is arbitrary, while  $\mathbf{K}$  denotes a time-reversal invariant momentum,  $2\mathbf{K} \equiv \mathbf{0}$ .

borrow for it, it is a nonequilibrium, or kinetic, magnetoelectric response [89]. [90] In particular, its time-even component may remain finite in a time-reversal-symmetric system [23, 89, 91], whereas an equilibrium magnetoelectric coefficient relating a time-even electric field to a time-odd magnetization is necessarily time odd.

The symmetry conditions follow directly from the defining representation. For non-coplanar order described by  $\Gamma$ , the time-odd and time-even components transform with spin indices in  $\Gamma$  and  $\Gamma \otimes \det \Gamma$ , and are therefore controlled by  $\Lambda^1 \Gamma$  and  $\Lambda^2 \Gamma$  respectively. Since the electric field transforms in the polar-vector representation  $\Gamma_1^-$ , the time-odd component is allowed by symmetry when

$$\alpha^- : \Gamma \cap \Gamma_1^- \neq 0. \quad (41)$$

The time-even component is allowed when

$$\alpha^+ : (\Gamma \otimes \det \Gamma) \cap \Gamma_1^- \neq 0. \quad (42)$$

For coplanar order the spin-only symmetry separates the two responses. The time-odd induced spin lies within the ordering plane and is controlled by  $\Gamma$ , whereas the time-even induced spin is perpendicular to the ordering plane and is controlled by  $\det \Gamma$  [91]. Thus

$$\alpha^- : \Gamma \cap \Gamma_1^- \neq 0, \quad (43a)$$

$$\alpha^+ : \det \Gamma \cap \Gamma_1^- \neq 0. \quad (43b)$$

For collinear order the spin-only symmetry forbids the time-even component entirely, so only the time-odd component survives, with the condition  $\Gamma \cap \Gamma_1^- \neq 0$ . In each case the condition on the time-odd component is the same as that for the equilibrium linear magnetoelectric effect, which shares its spatial tensor structure.

We summarize the conditions for collinear, coplanar and non-coplanar order as

**Time-Odd Edelstein Effect:** A spin space group defined by a real representation  $\Gamma$  allows a time-odd Edelstein response if and only if  $\Gamma$  has nonzero overlap with the polar-vector representation  $\Gamma_1^-$ .

**Time-Even Edelstein Effect:** A spin space group defined by a real representation  $\Gamma$  allows a time-even Edelstein response if and only if  $\Lambda^2 \Gamma$  has nonzero overlap with the polar-vector representation  $\Gamma_1^-$ .

The wave-vector content enters exactly as it does for the spin conductivities of Section IV C 1. Since  $\Gamma_1^-$  lives at  $\mathbf{k} = \mathbf{0}$ , the time-odd component is controlled by  $\Lambda^1 \Gamma$  and requires a  $\mathbf{k} = \mathbf{0}$  component of  $\Gamma$ , so a representation built entirely from finite wave vectors permits no time-odd Edelstein response. The time-even component is controlled by  $\Lambda^2 \Gamma$  and requires only a pair of wave vectors that cancel.

The condition for the time-even Edelstein response is precisely the rank-one case of the odd-in-momentum spin-texture criterion derived in Section IV D. For coplanar or non-coplanar order both require  $\Lambda^2 \Gamma$  to overlap with  $\Gamma_1^-$ . The symmetry conditions for the two effects are therefore identical: a  $p$ -wave momentum-space spin texture is allowed if and only if a time-even linear Edelstein response is allowed. The forward direction of this equivalence underlies the large non-relativistic Edelstein response found in nodal  $p$ -wave magnets [92].

This gives a direct transport interpretation of the lowest-rank odd-parity spin texture. Of the odd-rank multipoles of  $\mathbf{s}(\mathbf{q})$ , only the  $p$ -wave one has the momentum dependence of a polar vector, the representation  $\Gamma_1^-$  of the electric field driving the Edelstein response. Symmetry thus ties a time-even spin polarization at linear order to the  $p$ -wave part of the texture, and not to its  $f$ -wave or higher components. Odd-parity magnets therefore divide into those with a  $p$ -wave component and those whose lowest odd rank is higher, a distinction we return to in Section VI C.

For collinear order the spin-only symmetry confines the time-odd response to the ordering axis and forbids the time-even one. As for the spin conductivity, the remaining response obeys a further restriction in simple models that does not follow from the transformation laws. In the usual Bloch description the carriers move in a fixed collinear exchange field, so their spin along  $\hat{\mathbf{N}}$  is conserved,

$$[H, \mathbf{s} \cdot \hat{\mathbf{N}}] = [H_E, \mathbf{s} \cdot \hat{\mathbf{N}}] = 0, \quad (44)$$

where  $\mathbf{s}$  is the carrier spin and  $H_E$  the coupling to the electric field. Its expectation value then cannot change to any order in the applied field, and the uniform Edelstein response vanishes [93]. This conservation is not protected by symmetry. The residual spin rotations about  $\hat{\mathbf{N}}$  conserve only the total spin of the carriers and the ordered moments together. Once the carriers can exchange spin angular momentum with the ordered moments, for example through electron-magnon scattering,

the carrier spin is no longer conserved. A collinear magnet with  $\Gamma \cap \Gamma_1^- \neq 0$  should then generically show a time-odd Edelstein response, just as an antisymmetric spin conductivity appears once the spin channels are coupled.

## V. EXAMPLES

We now illustrate the representation-first approach with several magnetic materials hosting collinear, coplanar, and non-coplanar orders. In each case we identify the parent space group and the real representation describing the magnetic structure, and then infer the allowed physical responses directly from its representation content. We include examples where the parent space group of the spin space group differs from the full crystallographic space group, as occurs when the order selects only part of a wave-vector star, so that each  $\mathbf{k}$ -domain has its own parent group. We note that these conditions are symmetry statements about the allowed tensor components and apply irrespective of the magnitude or microscopic origin of the corresponding observables. For example, in insulating materials they may be strongly suppressed, particularly at low temperature, but are subject to the same symmetry constraints. A summary of these examples and the observables allowed in each is given in Table VI.

### A. $\text{MnF}_2$

The material  $\text{MnF}_2$  is a canonical example of an altermagnet [6, 7]. It crystallizes in the tetragonal rutile structure with space group  $P4_2/mnm$  (#136) [95], and has a collinear antiferromagnetic order [96] that appears below the Néel temperature  $T_N \approx 67$  K [97].

Each symmetry element in the space group is associated with a spin-space operation that either preserves or flips the spin direction and thus the parent space group of the ordered phase is the same as the crystallographic space group. The two Mn ions occupy the two sites of the  $2a$  orbit,  $(0, 0, 0)$  and  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ , so the one-dimensional representation follows from whether each generator preserves or exchanges them. Since  $P4_2/mnm$  is non-symmorphic these must be specified as full Seitz elements: the  $4_2$  screw  $[C_{4z}|\frac{1}{2}\frac{1}{2}\frac{1}{2}]$  and the two-fold screws  $[C_{2x}|\frac{1}{2}\frac{1}{2}\frac{1}{2}]$  about the in-plane  $x$  and  $y$  axes exchange the two sublattices, whereas the two-fold rotations  $[C_{2[110]}|\mathbf{0}]$  about the in-plane diagonals and the inversion  $[I|\mathbf{0}]$  preserve them (Fig. 3). Hence

$$\Gamma([C_{4z}|\frac{1}{2}\frac{1}{2}\frac{1}{2}]) = -1, \quad (45a)$$

$$\Gamma([C_{2x}|\frac{1}{2}\frac{1}{2}\frac{1}{2}]) = -1, \quad (45b)$$

$$\Gamma([C_{2[110]}|\mathbf{0}]) = +1, \quad (45c)$$

$$\Gamma([I|\mathbf{0}]) = +1, \quad (45d)$$

which identify the representation as

$$\Gamma = B_{2g} \quad (46)$$

of  $D_{4h}$ , using the convention in which  $B_{1g}$  and  $B_{2g}$  transform as  $x^2 - y^2$  and  $xy$ , respectively. Alternatively, we can obtain

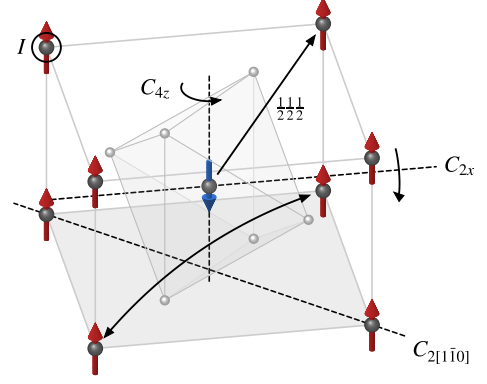


FIG. 3. The unit cell of  $\text{MnF}_2$ , with the  $\text{MnF}_6$  octahedron about the center shaded. The corner (red) and center (blue) Mn carry opposite moments along  $c$ . The screws  $[C_{4z}|\frac{1}{2}\frac{1}{2}\frac{1}{2}]$  and  $[C_{2x}|\frac{1}{2}\frac{1}{2}\frac{1}{2}]$  take the central Mn to a corner, so  $\Gamma = -1$ . The two-fold  $C_{2[110]}$  about the base diagonal exchanges two corners, and the inversion  $I$  through the ringed corner Mn fixes it, so  $\Gamma = +1$  for both.

the same result using the permutation representation of the two Mn sites, which is  $A_{1g} \oplus B_{2g}$  [98, 99], the  $A_{1g}$  being the trivial ferromagnet.

Since  $\Gamma = B_{2g}$  does not contain the trivial irrep  $A_{1g}$ , the order is compensated. The quadrupole representation in the  $D_{4h}$  point group is  $\Gamma_2^+ = A_{1g} \oplus B_{1g} \oplus B_{2g} \oplus E_g$ , so that  $\Gamma$  overlaps with  $\Gamma_2^+$  and piezomagnetism is allowed in the non-relativistic limit [13]. The overlap is with the single  $B_{2g}$  component of  $\Gamma_2^+$ , corresponding to the shear strain  $\epsilon_{xy}$ , so that the allowed coupling is  $\mathbf{M} \propto \epsilon_{xy} \hat{\mathbf{N}}$ : a basal-plane shear generates a moment along the Néel axis. This component was observed in  $\text{MnF}_2$  by Baruchel *et al.* [100] indirectly through domain selection in the presence of uniaxial strain. Earlier measurements by Borovik-Romanov observed a different transverse piezomagnetic component in  $\text{MnF}_2$ , which is absent in the strict non-relativistic limit [17]. The  $d$ -wave spin splitting and the relativistic transverse piezomagnetic response of  $\text{MnF}_2$  have also been discussed in terms of ferroically ordered magnetic octupoles [101]. The same  $B_{2g}$  invariant, read as a coupling between strain and magnetic field, underlies the elastocaloric response recently used to probe the multipolar character of the order in  $\text{MnF}_2$  [102]. The time-odd spin conductivity is allowed since  $\Gamma$  overlaps with  $\Gamma_2^+$ , as for the isostructural  $\text{RuO}_2$  [19]. The time-even spin conductivity and the anomalous Hall effect vanish since the order is collinear. The same representation fixes the momentum-space spin texture. Since  $\Gamma = B_{2g}$  occurs in the quadrupole representation, the leading texture is  $d$ -wave, with  $s(\mathbf{q})$  having the same symmetry as  $q_x q_y \hat{\mathbf{N}}$  in the convention used here [Fig. 2(a)] [103].

### B. $\text{MnTe}$

$\text{MnTe}$  is also a prominent candidate altermagnetic material. It crystallizes in the NiAs structure with space group  $P6_3/mmc$

| Compound                                                                                            | Order        | Crystal               |                                 | Parent              |                       | Exterior powers   |                   |                   | Allowed observables |              |                         |              |                               | Texture |
|-----------------------------------------------------------------------------------------------------|--------------|-----------------------|---------------------------------|---------------------|-----------------------|-------------------|-------------------|-------------------|---------------------|--------------|-------------------------|--------------|-------------------------------|---------|
|                                                                                                     |              | Space group ( $G_0$ ) | $\mathbf{k}$                    | Space group ( $G$ ) | $\mathbf{k}$          | $\Lambda^1\Gamma$ | $\Lambda^2\Gamma$ | $\Lambda^3\Gamma$ | $\mathbf{M}$        | $\mathbf{C}$ | $\Sigma^{-/+}$          | $\sigma^-$   | $\alpha^{-/+}$                |         |
| MnF <sub>2</sub>                                                                                    | collinear    | $P4_2/mnm$ (#136)     | $\mathbf{0}$                    | $P4_2/mnm$ (#136)   | $\mathbf{0}$          | $B_{2g}$          | 0                 | 0                 | $\times$            | $\checkmark$ | $\checkmark/\times$     | $\times$     | $\times/\times$               | $d$     |
| MnTe                                                                                                | collinear    | $P6_3/mmc$ (#194)     | $\mathbf{0}$                    | $P6_3/mmc$ (#194)   | $\mathbf{0}$          | $B_{1g}$          | 0                 | 0                 | $\times$            | $\times$     | $\times/\times$         | $\times$     | $\times/\times$               | $g$     |
| Mn <sub>3</sub> Ge <sup>‡</sup>                                                                     | coplanar     | $P6_3/mmc$ (#194)     | $\mathbf{0}$                    | $P6_3/mmc$ (#194)   | $\mathbf{0}$          | $E_{2g}$          | $A_{2g}$          | 0                 | $\times$            | $\checkmark$ | $\checkmark/\checkmark$ | $\times$     | $\times/\times$               | $d$     |
| FeF <sub>3</sub>                                                                                    | non-coplanar | $Fd\bar{3}m$ (#227)   | $\mathbf{0}$                    | $Fd\bar{3}m$ (#227) | $\mathbf{0}$          | $T_{2g}$          | $T_{1g}$          | $A_{2g}$          | $\times$            | $\checkmark$ | $\checkmark/\checkmark$ | $\times$     | $\times/\times$               | $d$     |
| CoTa <sub>3</sub> S <sub>6</sub> <sup>‡</sup>                                                       | non-coplanar | $P6_322$ (#182)       | $\mathbf{M}$                    | $P6_322$ (#182)     | $\mathbf{M}$          | $^*M_1$           | $^*M_2$           | $A_2$             | $\times$            | $\times$     | $\times/\times$         | $\checkmark$ | $\times/\times$               | —       |
| Cr <sub>2</sub> O <sub>3</sub>                                                                      | collinear    | $R\bar{3}c$ (#167)    | $\mathbf{0}$                    | $R\bar{3}c$ (#167)  | $\mathbf{0}$          | $A_{2u}$          | 0                 | 0                 | $\times$            | $\times$     | $\times/\times$         | $\times$     | $\checkmark^{\dagger}/\times$ | —       |
| MnO                                                                                                 | collinear    | $Fm\bar{3}m$ (#225)   | $\mathbf{L}$                    | $R\bar{3}m$ (#166)  | $\mathbf{T}$          | $T_1^+$           | 0                 | 0                 | $\times$            | $\times$     | $\times/\times$         | $\times$     | $\times/\times$               | —       |
| Gd <sub>3</sub> (Ru <sub>0.95</sub> Rh <sub>0.05</sub> ) <sub>4</sub> Al <sub>12</sub> <sup>‡</sup> | coplanar     | $P6_3/mmc$ (#194)     | $(\frac{1}{6}, \frac{1}{6}, 0)$ | $Cmcm$ (#63)        | $(\frac{1}{3}, 0, 0)$ | $^*\Sigma_1$      | $B_{3u}$          | 0                 | $\times$            | $\times$     | $\times/\times$         | $\times$     | $\times/\checkmark$           | $p$     |

TABLE VI. Summary of the examples of Sections V A to V H, listed in the order in which they are discussed. A double dagger marks compounds treated in an idealized form of the reported magnetic structure, as described in the corresponding section. For each compound we list the type of magnetic order, the crystallographic space group, the parent space group  $G$  of the spin space group, the propagation vector  $\mathbf{k}$  referred to each of the two groups in turn, the exterior powers of the real representation  $\Gamma = \Lambda^1\Gamma$  of  $G$  that defines the spin space group, with a zero where a power vanishes by the spin-only symmetry. The next five columns indicate whether the net magnetization  $\mathbf{M}$ , the piezomagnetic coefficient  $\mathbf{C}_{\mu\nu}$ , the spin conductivities  $\Sigma^{-/+}$ , the anomalous Hall conductivity  $\sigma^-$ , and the Edelstein responses  $\alpha^{-/+}$  are allowed ( $\checkmark$ ) or forbidden ( $\times$ ) by symmetry in the non-relativistic limit. The paired  $\Sigma^{-/+}$  and  $\alpha^{-/+}$  entries give the time-odd and time-even components in that order. A dagger marks an entry allowed by the representation overlap that nonetheless vanishes when the carrier spin is conserved, Eq. (44), as for Bloch electrons in a fixed collinear exchange field. The last column gives the leading momentum-space spin texture, labeled by the lowest multipole rank allowed by the conditions of Table V, with a dash where the texture vanishes identically. The finite- $\mathbf{k}$  examples are labeled in the Miller-Love notation [94]. Propagation vectors are given in the conventional reciprocal basis of the group heading their column, and are quoted by their Brillouin-zone label where one applies.

(#194) [104], and has a collinear antiferromagnetic order in which the moments are ferromagnetically aligned within each triangular Mn layer and alternate in sign between neighboring layers along the  $c$  axis [105, 106], appearing below the Néel temperature  $T_N \approx 307$  K [107]. It is a leading example of a  $g$ -wave altermagnet [108]. The Mn ions occupy the two sites of the  $2a$  orbit, at  $z = 0$  and  $z = 1/2$ , so the one-dimensional representation describing the order can be identified directly from how the space-group operations act on these two magnetic sublattices.

As for  $P4_2/mnm$ , the space group  $P6_3/mmc$  is non-symmorphic and its elements must be specified as full Seitz elements. The  $6_3$  screw  $[C_{6z}|00\frac{1}{2}]$  maps the Mn layer at  $z = 0$  onto the layer at  $z = 1/2$  and therefore exchanges the two oppositely polarized sublattices. Consequently,

$$\Gamma([C_{6z}|00\frac{1}{2}]) = -1. \quad (47)$$

The inversion  $[I|\mathbf{0}]$ , on the other hand, maps each Mn layer back onto itself, since  $z = 0$  is fixed and  $z = 1/2$  is mapped to  $-1/2 \equiv 1/2$ , and hence

$$\Gamma([I|\mathbf{0}]) = +1. \quad (48)$$

Finally, the in-plane two-folds fall into two classes distinguished by their fractional translations (Fig. 4). Those about the  $a$  axes,  $[C_{2[100]}|\mathbf{0}]$ , carry no translation and map  $z = 0$  and  $z = 1/2$  each onto itself, whereas those about the intervening in-plane axes,  $[C_{2[1\bar{1}0]}|00\frac{1}{2}]$ , rotated by  $30^\circ$  from the first set, exchange the two sublattices. Taking the former as the  $C_2'$  class [109], we have

$$\Gamma([C_{2[100]}|\mathbf{0}]) = +1. \quad (49)$$

These characters identify the one-dimensional representation of  $D_{6h}$  as

$$\Gamma = B_{1g}. \quad (50)$$

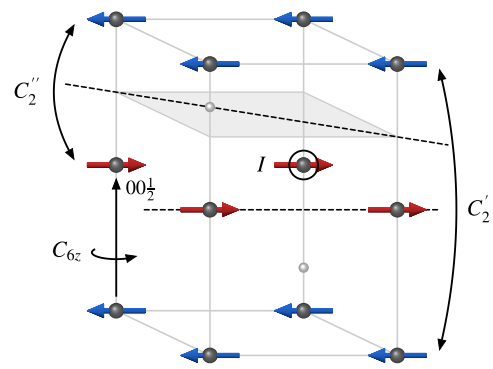


FIG. 4. The unit cell of MnTe from  $z = -\frac{1}{2}$  to  $\frac{1}{2}$ , with Mn dark and Te light. The Mn layers are ferromagnetic and alternate between the two sublattices, red at  $z = 0$  and blue at  $z = \pm\frac{1}{2}$ . The two-fold  $C_2'$  about  $a$  in the red layer maps each layer onto itself, so  $\Gamma = +1$ . The two-fold  $C_2''$  about  $[1\bar{1}0]$  in the shaded plane  $z = \frac{1}{4}$  exchanges red and blue layers, so  $\Gamma = -1$ . The screw  $[C_{6z}|00\frac{1}{2}]$  about the lower-left Mn column takes the blue Mn to the red one above it, so  $\Gamma = -1$ . The inversion  $I$  through the ringed Mn fixes the red layer and exchanges the two blue ones, so  $\Gamma = +1$ .

Alternatively, we can obtain the same result using the permutation representation of the two Mn sites, which is  $A_{1g} \oplus B_{1g}$  [98, 99].

Since  $\Gamma = B_{1g}$  does not contain the trivial irrep  $A_{1g}$ , the order is compensated. In the  $D_{6h}$  point group the axial-vector and quadrupole representations are

$$\Gamma_1^+ = A_{2g} \oplus E_{1g}, \quad (51a)$$

$$\Gamma_2^+ = A_{1g} \oplus E_{1g} \oplus E_{2g}. \quad (51b)$$

Thus  $B_{1g}$  has no overlap with either  $\Gamma_1^+$  or  $\Gamma_2^+$ , and there is

no linear piezomagnetism or time-odd spin conductivity in the non-relativistic limit. The time-even spin conductivity and anomalous Hall conductivity vanish because the order is collinear; the spontaneous anomalous Hall effect measured in MnTe [110] thus requires relativistic interactions or extrinsic perturbations that break the protecting spin-only symmetry. The absence of overlap with the scalar, axial-vector, and quadrupole representations is the representation-theory signature of the  $g$ -wave character of MnTe: the order first transforms like a higher-rank spatial multipole. By the conditions of Section IV D the momentum-space texture is correspondingly  $g$ -wave. With the present choice of crystallographic axes, the leading form of  $s(\mathbf{q})$  transforms as  $q_y q_z (3q_x^2 - q_y^2) \hat{\mathbf{N}}$ , the  $B_{1g}$  representation defining the order.

### C. Mn<sub>3</sub>Ge

The material Mn<sub>3</sub>Ge, in its hexagonal  $D0_{19}$  form, crystallizes in space group  $P6_3/mmc$  (#194) [111, 112] and hosts a coplanar 120° antiferromagnetic order at  $\mathbf{k} = 0$  [113–115], appearing below the Néel temperature  $T_N \approx 365$  K [112]. The parent space group of the ordered phase is the same as the crystallographic space group. We have assumed an idealized coplanar 120° structure with equal moments; the observed state carries in addition a small weak-ferromagnetic moment, an inevitable consequence of its symmetry once relativistic effects are included [115].

The two-dimensional representation describing the (idealized) order can be identified directly from the action of the space-group generators on the magnetic structure. The inversion  $[I|\mathbf{0}]$  exchanges the two kagome layers and relates sites with parallel moments, so that  $\Gamma([I|\mathbf{0}]) = \mathbf{1}$  and the representation is even under inversion. The three-fold rotation  $[C_{3z}|\mathbf{0}]$  cyclically permutes the three Mn sublattices within each layer, and since the moments on successive sublattices differ by an angle of  $2\pi/3$ ,  $\Gamma([C_{3z}|\mathbf{0}])$  is a spin rotation through  $\mp 2\pi/3$ , with trace  $-1$ . The  $6_3$  screw  $[C_{6z}|00\frac{1}{2}]$  maps the kagome layer at  $z = 1/4$  onto that at  $z = 3/4$  and permutes the three moment directions in the opposite cyclic order (Fig. 5), so  $\Gamma([C_{6z}|00\frac{1}{2}]) = \Gamma([C_{3z}|\mathbf{0}])^{-1}$  is a spin rotation through  $\pm 2\pi/3$ , and its trace is also  $-1$ . Together with the even inversion parity, the character under the six-fold operation distinguishes the representation as

$$\Gamma = E_{2g}, \quad (52)$$

rather than  $E_{1g}$ , for which  $\text{tr } \Gamma([C_{6z}|00\frac{1}{2}]) = +1$ . [116] We note that the “triangular” and “inverse triangular” 120° states (distinguished by the sign of their vector spin chirality), which carry distinct magnetic irreps once relativistic effects are included, differ only by a global spin rotation and therefore share the same  $\Gamma$ .

Since  $\Gamma = E_{2g}$  does not contain the trivial irrep  $A_{1g}$ , the order is compensated. The quadrupole representation in the  $D_{6h}$  point group is

$$\Gamma_2^+ = A_{1g} \oplus E_{1g} \oplus E_{2g}, \quad (53)$$

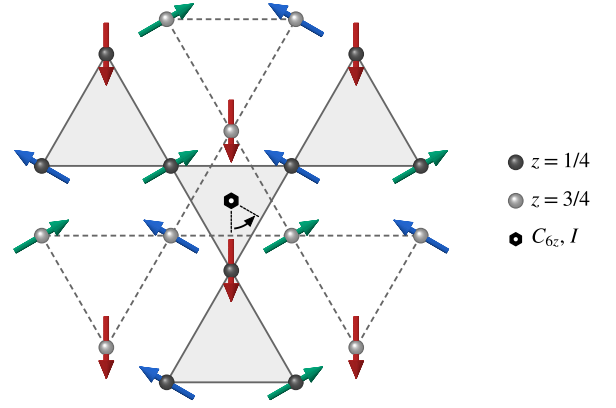


FIG. 5. The 120° order of Mn<sub>3</sub>Ge viewed along  $c$ . Inversion through the center maps each site onto one with the same moment, so  $\Gamma([I|\mathbf{0}]) = \mathbf{1}$ . The screw  $[C_{6z}|00\frac{1}{2}]$  carries each site onto the next one around the center (arc), whose moment is rotated by 120°, so  $\Gamma([C_{6z}|00\frac{1}{2}])$  is a 120° spin rotation.

so that  $\Gamma$  overlaps with  $\Gamma_2^+$  and the order exhibits piezomagnetism. This has been observed in the isostructural Mn<sub>3</sub>Sn, where the strain-induced moment has been used to switch the anomalous Hall effect at room temperature [117]. The time-odd spin conductivity is allowed for the same reason, consistent with previous analyses of spin-polarized currents in non-collinear antiferromagnets [53]. For the time-even spin conductivity, the relevant representation is

$$\Lambda^2 \Gamma = \det \Gamma = A_{2g}. \quad (54)$$

Since  $A_{2g}$  overlaps with the axial-vector representation  $\Gamma_1^+$ , the time-even spin conductivity is also allowed. Related time-even spin-current responses have been discussed previously in non-collinear antiferromagnets [54]. The anomalous Hall conductivity vanishes in the non-relativistic limit because the order is coplanar; the large anomalous Hall effect observed in Mn<sub>3</sub>Ge [118, 119] is thus necessarily of relativistic origin. The spin texture lies in the ordering plane and is  $d$ -wave. Choosing a suitable basis  $\hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2$  of the ordering plane, the leading form of  $s(\mathbf{q})$  transforms as  $(q_x^2 - q_y^2) \hat{\mathbf{N}}_1 + 2q_x q_y \hat{\mathbf{N}}_2$  [Fig. 2(b)]; the perpendicular odd component vanishes since  $\det \Gamma = A_{2g}$  is even under inversion.

### D. Pyrochlore FeF<sub>3</sub>

The material FeF<sub>3</sub>, in its metastable pyrochlore form, crystallizes in space group  $Fd\bar{3}m$  (#227), in which the Fe<sup>3+</sup> ions form a three-dimensional network of corner-sharing tetrahedra [120]. Geometrical frustration suppresses long-range order to  $T_N \approx 15.5$  K [121], below which the moments adopt a  $\mathbf{k} = 0$  non-collinear all-in-all-out (AIAO) structure [122]; the selection of this state by a positive biquadratic term has been studied microscopically [123]. The four moments on each

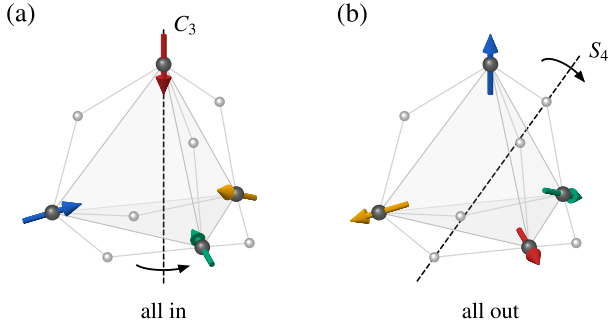


FIG. 6. One  $\text{Fe}_4$  tetrahedron of pyrochlore  $\text{FeF}_3$ , with an F bridging each edge. (a) The all-in order is unchanged by the three-fold rotation about the axis through one Fe, so  $C_3$  has character +1. (b) Its image under the improper rotation  $S_4$  about  $z$ , each moment moved with its site and turned as an axial vector (colors follow the sites), is all-out. Since  $S_4 = IC_4^{-1}$  and inversion leaves the order unchanged,  $C_4$  has character -1.

tetrahedron point along the four local  $\langle 111 \rangle$  axes,

$$\begin{aligned} \hat{n}_0 &= \frac{1}{\sqrt{3}}(+1, +1, +1), & \hat{n}_1 &= \frac{1}{\sqrt{3}}(+1, -1, -1), \\ \hat{n}_2 &= \frac{1}{\sqrt{3}}(-1, +1, -1), & \hat{n}_3 &= \frac{1}{\sqrt{3}}(-1, -1, +1), \end{aligned} \quad (55)$$

with the two magnetic domains corresponding to  $\mathbf{S}_a = \pm \mathbf{S} \hat{n}_a$ .

It is useful first to identify the conventional magnetic representation of this state, in which spatial and spin rotations act together. A three-fold rotation about a  $\langle 111 \rangle$  axis preserves the AIAO pattern [Fig. 6(a)]: it fixes one local axis and cyclically permutes the other three. By contrast, a four-fold rotation about a cubic axis reverses the pattern [Fig. 6(b)]. For example, under  $C_{4z}$ ,

$$\hat{n}_0 \rightarrow -\hat{n}_1, \quad \hat{n}_1 \rightarrow -\hat{n}_3, \quad \hat{n}_3 \rightarrow -\hat{n}_2, \quad \hat{n}_2 \rightarrow -\hat{n}_0,$$

so that, together with the corresponding permutation of the four Fe sublattices, the two AIAO domains are exchanged. Inversion leaves the magnetic pattern unchanged, since the Fe magnetic sublattices are mapped onto themselves modulo a Bravais translation and the magnetic moments are axial vectors. The corresponding one-dimensional order parameter therefore has characters

$$\chi(C_3) = +1, \quad \chi(C_4) = -1, \quad \chi(I) = +1, \quad (56)$$

which identify the conventional AIAO magnetic representation as  $A_{2g}$  of  $O_h$ .

The representation relevant for the spin space group is not this one-dimensional magnetic representation. In the conventional description the spin transforms as an axial vector, which carries the representation

$$\Gamma_1^+ = T_{1g}. \quad (57)$$

Interpreting this as a spin space group, as for any magnetic space group (Section VIA), we have

$$\Gamma = A_{2g} \otimes T_{1g} = T_{2g}. \quad (58)$$

Since  $\Gamma = T_{2g}$  contains no copy of the trivial irrep  $A_{1g}$ , the order is compensated. The quadrupole representation in the  $O_h$  point group is

$$\Gamma_2^+ = E_g \oplus T_{2g}, \quad (59)$$

so that  $\Gamma$  overlaps with  $\Gamma_2^+$  and the order exhibits piezomagnetism. The time-odd spin conductivity is allowed for the same reason.

The higher exterior powers follow from  $\Gamma = A_{2g} \otimes T_{1g}$ . The axial-vector representation has unit determinant for every spatial operation, and multiplying a three-dimensional representation by the one-dimensional  $A_{2g}$  multiplies its determinant by the cube of that character, and hence by the character itself, so that

$$\Lambda^3 \Gamma = \det \Gamma = A_{2g}, \quad (60a)$$

$$\Lambda^2 \Gamma = \Gamma \otimes \det \Gamma = T_{2g} \otimes A_{2g} = T_{1g}. \quad (60b)$$

The time-even spin conductivity is therefore allowed, since  $\Lambda^2 \Gamma = T_{1g}$  overlaps with the axial-vector representation  $\Gamma_1^+$ . The anomalous Hall conductivity vanishes in the non-relativistic limit, since  $\Lambda^3 \Gamma = A_{2g}$  does not, as required for any cubic parent point group (Section IVC 2). In the spin frame used above, the conductivity tensors take the particularly simple forms

$$\Sigma_{\nu\rho}^{\mu,-} \propto |\varepsilon_{\mu\nu\rho}|, \quad \Sigma_{\nu\rho}^{\mu,+} \propto \varepsilon_{\mu\nu\rho},$$

where we adopt  $\varepsilon_{xyz} = +1$ . Thus the time-odd conductivity is symmetric in its spatial indices, while the time-even conductivity is antisymmetric, with one independent coefficient in each channel. Pyrochlore  $\text{FeF}_3$  thus provides a simple  $\mathbf{k} = 0$  non-coplanar example in which both time-even and time-odd spin conductivities are allowed while the anomalous Hall conductivity is forbidden.

The leading momentum-space spin texture is a three-component  $d$ -wave. Up to a global choice of spin frame  $\hat{N}_1, \hat{N}_2, \hat{N}_3$  the leading form of  $s(\mathbf{q})$  transforms as  $q_y q_z \hat{N}_1 + q_z q_x \hat{N}_2 + q_x q_y \hat{N}_3$ , the standard  $T_{2g}$  quadrupole [Fig. 2(c)]. Odd-rank contributions are forbidden since  $\Gamma \otimes \det \Gamma = T_{1g}$  is even under inversion.

### E. $\text{CoTa}_3\text{S}_6$

The material  $\text{CoTa}_3\text{S}_6$  crystallizes in space group  $P6_322$  (#182), with the Co ions forming triangular layers [124]. The low-temperature magnetic state, below  $T_{N2} \approx 26.5$  K [124], is a non-coplanar tetrahedral triple- $Q$  order involving the three symmetry-related  $M$  points [125, 126]

$$\mathbf{Q}_1 = (\frac{1}{2}, 0, 0), \quad \mathbf{Q}_2 = (0, \frac{1}{2}, 0), \quad \mathbf{Q}_3 = (\frac{1}{2}, -\frac{1}{2}, 0). \quad (61)$$

Choosing three orthogonal spin directions  $\hat{N}_1, \hat{N}_2$ , and  $\hat{N}_3$ , the tetrahedral state may be written in the form

$$\mathbf{S}(\mathbf{r}) = \frac{S}{\sqrt{3}} \sum_{n=1}^3 \hat{N}_n \cos(\mathbf{Q}_n \cdot \mathbf{r}), \quad (62)$$

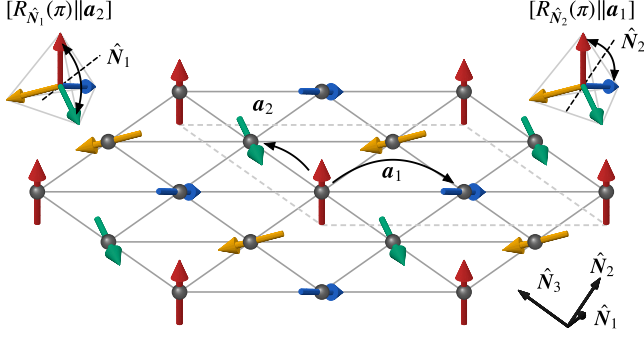


FIG. 7. The idealized tetrahedral triple- $Q$  order of  $\text{CoTa}_3\text{S}_6$ , Eq. (62), on one triangular Co layer, with its four moment directions in the insets and the spin frame  $\hat{N}_1, \hat{N}_2, \hat{N}_3$  at lower right. Translation by  $\mathbf{a}_1$  takes each site to one whose moment differs by the spin rotation  $R_{\hat{N}_2}(\pi)$ , so the order is invariant under  $[R_{\hat{N}_2}(\pi)||\mathbf{a}_1]$  (right inset), giving  $\Gamma([\mathbf{1}|\mathbf{a}_1]) = \text{diag}(-1, +1, -1)$ . Likewise  $[R_{\hat{N}_1}(\pi)||\mathbf{a}_2]$  (left inset) gives  $\Gamma([\mathbf{1}|\mathbf{a}_2]) = \text{diag}(+1, -1, -1)$ . The gray dashed outline marks the  $2 \times 2$  magnetic cell.

with the origin chosen on a Co site. The second Co layer of the unit cell follows from the first by the  $6_3$  screw, with scalar spin chirality of the same sign in both layers, the ferro-chiral stacking argued for by Park *et al.* [125]. We use the equal-amplitude tetrahedral state of Eq. (62) as an ideal representative. Recent work reports coexisting chirality and nematicity in the low-temperature, low-field phase, consistent with non-equilateral triple- $Q$  order [127]; we return to such distortions below.

Since each  $\mathbf{Q}_n$  is a time-reversal invariant  $M$  point, translations act on the three spin components only by signs. For primitive translations  $\mathbf{a}_1$  and  $\mathbf{a}_2$ , a convenient choice of basis gives

$$\Gamma([\mathbf{1}|\mathbf{a}_1]) = \text{diag}(-1, +1, -1), \quad (63a)$$

$$\Gamma([\mathbf{1}|\mathbf{a}_2]) = \text{diag}(+1, -1, -1). \quad (63b)$$

The four possible sign patterns generated by these translations give the four spin directions

$$(+, +, +), \quad (-, +, -), \quad (+, -, -), \quad (-, -, +), \quad (64)$$

which are the vertices of a regular tetrahedron (Fig. 7). This is a version of the tetrahedral triple- $Q$  order discussed in Refs. [33, 67] on the two-dimensional triangular lattice.

The point-group operations act by permuting the three arms of the  $M$ -point star and hence simultaneously permuting the three spin directions  $\hat{N}_n$ . In particular, the six-fold operation cyclically permutes the three wave vectors, while an in-plane two-fold operation exchanges two of them. Together with the translation matrices above, this defines the real three-dimensional *star* representation associated with the three  $M$  points, rather than a three-dimensional representation at a single wave vector. In the conventional Miller–Love notation [94] it is  $^*M_1$ , the representation induced from the one-dimensional little-group irrep  $M_1$  at  $\mathbf{k} = M$ . For this symmetric triple- $Q$  state, permutations of the three equally weighted star arms are implemented by orthogonal transformations in spin space,

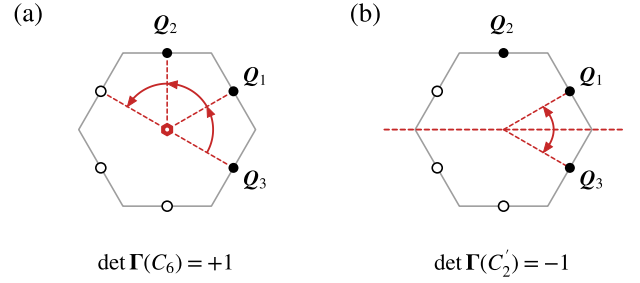


FIG. 8. The  $M$ -point arms  $\mathbf{Q}_n$  of  $\text{CoTa}_3\text{S}_6$  (filled) and their equivalent opposites (open). For the ferro-chiral stacking,  $\det \Gamma$  is the sign of the permutation of the arms. (a) The six-fold rotation cycles them, so  $\det \Gamma(C_6) = +1$ . (b) The two-fold about the dashed axis exchanges  $\mathbf{Q}_1$  and  $\mathbf{Q}_3$  and takes  $\mathbf{Q}_2$  to its opposite, so  $\det \Gamma(C_2') = -1$ .

allowing the full crystallographic space group to be retained as the parent, in contrast to the single- $\mathbf{k}$  orders of  $\text{MnO}$  and  $\text{Gd}_3(\text{Ru}_{0.95}\text{Rh}_{0.05})_4\text{Al}_{12}$  below.

Its determinant can be fixed without constructing the full representation matrices: for this stacking,  $\det \Gamma(g)$  is simply the parity of the permutation of the three arms by  $g$  (Fig. 8). Translations permute no arms, and indeed have determinant  $+1$ , since

$$\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 \equiv \mathbf{0} \quad (65)$$

modulo a reciprocal lattice vector. An operation that keeps each Co layer in place can be written as a symmetry fixing the Co site chosen as the origin, followed by a lattice translation. The site symmetry simply permutes the three cosine modes, and hence the  $\hat{N}_n$ , while the translation has determinant  $+1$ . The determinant of the combined operation is therefore the parity of the arm permutation. Every other operation is one of these composed with the  $6_3$  screw, which permutes the arms cyclically, so the rule holds throughout provided  $\det \Gamma(6_3) = +1$ . This last sign is fixed by the stacking. The  $6_3$  screw is a rotation about  $\hat{c}$  that carries each triangle of one layer to a triangle of the other with the same orientation. Since the scalar chirality transforms with  $\det \Gamma(g)$  [Eq. (10)],  $\det \Gamma(6_3) = +1$  exactly when the two layers have chirality of the same sign, as here. A six-fold rotation, a three-cycle, therefore has determinant  $+1$ , whereas an in-plane two-fold rotation exchanges two arms and has determinant  $-1$ . These are precisely the characters of  $A_2$  in the  $D_6$  point group, and hence

$$\Lambda^3 \Gamma = \det \Gamma = A_2. \quad (66)$$

Since  $\Gamma$  is built entirely from finite-wave-vector components at the  $M$  point, the order is compensated, with no linear piezomagnetism or time-odd spin conductivity (Section IV A). For the time-even spin conductivity the relevant representation is  $\Lambda^2 \Gamma = \Gamma \otimes \det \Gamma = ^*M_2$ . Since  $\det \Gamma = A_2$  is a  $\mathbf{k} = 0$  representation, tensoring by it does not change the wave-vector content of  $\Gamma$ , and therefore the time-even spin conductivity vanishes as well.

The anomalous Hall conductivity however is different. The order is non-coplanar, so it is not forbidden by a spin-only sym-

metry, and its symmetry is controlled by the top exterior power  $\Lambda^3 \Gamma = \det \Gamma$ . In  $D_6$  the  $A_2$  irrep is the out-of-plane component of the axial-vector representation. Since  $\det \Gamma = A_2$ , an anomalous Hall conductivity in the plane, or equivalently a Hall pseudovector along the  $c$  axis, is allowed even in the non-relativistic limit, consistent with the large spontaneous Hall conductivity  $\sigma_{xy}$  measured in the triple- $Q$  phase [125, 126]. In contrast, the momentum-space spin texture vanishes identically. The representation  $\Gamma$  contains only the finite- $M$  components, while  $\Lambda^2 \Gamma = \Gamma \otimes \det \Gamma$  has the same wave-vector content, since  $\det \Gamma = A_2$  lives at  $\mathbf{k} = 0$ , so neither even- nor odd-rank momentum multipoles are allowed despite the finite anomalous Hall response. This allowed Hall response and the vanishing spin magnetization and spin conductivities were also obtained from spin-group symmetry by Watanabe *et al.* [12], who noted that the Hall response occurs without non-relativistic spin-momentum locking.

The precise stacking matters only for the Hall response. Reversing every moment of the second layer gives a tetrahedral order with the same parent space group, but with scalar chirality of opposite sign in the two layers. This reverses  $\Gamma(g)$  for every operation that exchanges the layers, replacing  $\Gamma$  by  $\Gamma \otimes B_2 = {}^*M_4$ , where  $B_2$  is the irrep of  $D_6$  that is  $-1$  on those operations. The exterior square  $\Lambda^2 \Gamma = {}^*M_2$  is unchanged, so this order too is compensated without spin conductivities or a spin texture. But since  $\det \Gamma = A_2 \otimes B_2 = B_1$  does not occur in  $\Gamma_1^+ = A_2 \oplus E_1$ , its anomalous Hall response vanishes. The measured  $\sigma_{xy}$  thus also selects the ferro-chiral stacking.

This separation is not specific to this tetrahedral order. More generally, for any three-dimensional  $\Gamma$  with no  $\mathbf{k} = 0$  component but with  $\det \Gamma$  trivial under translations,  $\Lambda^2 \Gamma = \Gamma \otimes \det \Gamma$  also has no  $\mathbf{k} = 0$  component. Both spin conductivities and the momentum-space spin texture therefore vanish, while an anomalous Hall response remains possible when  $\det \Gamma$  overlaps with  $\Gamma_1^+$ . In particular, these translation-enforced prohibitions do not require equal amplitudes. If the three components in Eq. (62) have unequal but nonzero amplitudes and remain mutually orthogonal, the translations still act as in Eqs. (63a) and (63b), so both spin conductivities and the momentum-space spin texture still vanish.

## F. $\text{Cr}_2\text{O}_3$

The material  $\text{Cr}_2\text{O}_3$  crystallizes in the corundum structure with space group  $R\bar{3}c$  (#167) [128], and has a collinear antiferromagnetic order at  $\mathbf{k} = 0$  [129, 130] below the Néel temperature  $T_N \approx 307$  K [131]. The Cr moments lie along the rhombohedral axis and alternate in sign between the two magnetic sublattices. The one-dimensional representation follows directly from how the generators act on these sublattices. A three-fold rotation about the rhombohedral axis preserves them, while the in-plane two-fold rotation and the inversion each exchange them. These two act on different pairs of Cr sites (Fig. 9): the two-fold axes of  $R\bar{3}c$  lie at  $z = \frac{1}{4}$  rather than through the origin, so  $C_2'$  relates  $(0, 0, z)$  to  $(0, 0, \frac{1}{2} - z)$ , the near neighbor across the shared octahedral face, whereas the

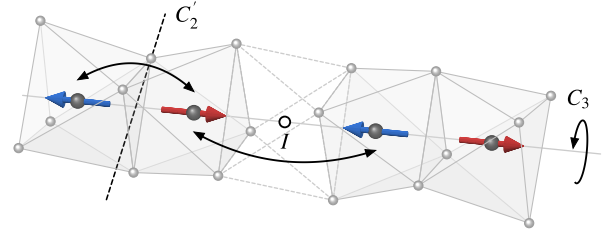


FIG. 9. A column of octahedra along the three-fold axis  $C_3$  of  $\text{Cr}_2\text{O}_3$ , drawn horizontally. Pairs of face-sharing  $\text{CrO}_6$  octahedra alternate with empty ones (dashed), and the moments alternate along the axis. The two-fold axis  $C_2'$  in the shared face exchanges the two Cr of a pair, and the inversion center  $I$  in the empty octahedron exchanges Cr across it. Both join opposite moments, so  $\Gamma(C_2') = \Gamma(I) = -1$ .

inversion relates it to  $(0, 0, -z)$ . Thus

$$\Gamma(C_3) = +1, \quad (67a)$$

$$\Gamma(C_2') = -1, \quad (67b)$$

$$\Gamma(I) = -1. \quad (67c)$$

These characters identify the representation as

$$\Gamma = A_{2u} \quad (68)$$

of the  $D_{3d}$  point group, with  $\Gamma(S_6) = -1$  and  $\Gamma(\sigma_d) = +1$  for the remaining improper operations.

Since  $\Gamma = A_{2u}$  does not contain the trivial irrep  $A_{1g}$ , the order is compensated. The quadrupole representation in the  $D_{3d}$  point group is

$$\Gamma_2^+ = A_{1g} \oplus 2E_g, \quad (69)$$

which has no overlap with  $A_{2u}$ . Thus there is no linear piezomagnetism in the non-relativistic limit. Similarly, the axial-vector representation is

$$\Gamma_1^+ = A_{2g} \oplus E_g, \quad (70)$$

so  $A_{2u}$  has no overlap with  $\Gamma_0^+$ ,  $\Gamma_1^+$ , or  $\Gamma_2^+$ . The time-odd spin conductivity therefore vanishes, while the time-even spin conductivity and anomalous Hall conductivity vanish because the order is collinear. Thus  $\text{Cr}_2\text{O}_3$  provides a simple example of a compensated collinear antiferromagnet that, unlike  $\text{MnF}_2$ , is not piezomagnetic in the non-relativistic limit. The momentum-space spin texture also vanishes in the non-relativistic limit. Collinearity requires it to be even in momentum, while  $\Gamma = A_{2u}$  is odd under inversion, so  $\Gamma$  cannot occur in any of the allowed even-rank momentum multipoles. The same inversion parity does permit the tensor structure of a time-odd Edelstein response: the polar-vector representation of  $D_{3d}$  is  $\Gamma_1^- = A_{2u} \oplus E_u$ , which contains  $\Gamma$ . This is the condition of Section IV E, and equivalently that for the equilibrium linear magnetoelectric effect, for which  $\text{Cr}_2\text{O}_3$  is the archetypal material [85, 86]. In the non-relativistic limit the uniform Edelstein response vanishes in simple models in which the carrier spin is conserved, such as Bloch electrons

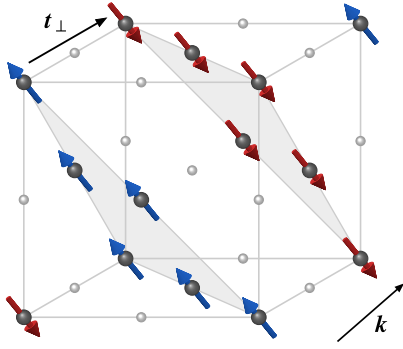


FIG. 10. The unit cell of MnO, with Mn dark and O light. The moments are ferromagnetic within (111) planes, two of which are shaded, and alternate between them. The propagation vector  $\mathbf{k} = L$  is normal to the planes, and a translation  $\mathbf{t}_\perp$  between neighboring planes reverses the moments, Eq. (74).

in a fixed collinear exchange field, but is generically allowed once the carriers can exchange spin with the ordered moments (Section IV E). The equilibrium magnetoelectric coefficient is not restricted in this way. For example, exchange striction and thermal spin fluctuations produce a finite longitudinal equilibrium magnetoelectric response in  $\text{Cr}_2\text{O}_3$  without spin-orbit coupling [132].

### G. MnO

The material MnO crystallizes in the rocksalt structure with space group  $Fm\bar{3}m$  (#225) [133] and develops the type-II collinear antiferromagnetic order [134–136] with propagation vector

$$\mathbf{k} = L = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right). \quad (71)$$

The magnetic structure (Fig. 10) consists of ferromagnetic (111) planes whose moments alternate in sign between neighboring planes below the Néel temperature  $T_N \approx 118$  K [135, 137], accompanied by a rhombohedral contraction along the selected  $\langle 111 \rangle$  axis [137].

Unlike the previous examples, the parent space group of a single  $\mathbf{k}$ -domain is not the full crystallographic space group. The cubic point group relates the four arms of the  $L$ -point star,

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \quad (72)$$

up to reciprocal lattice vectors. A single- $Q$  domain selects one of these arms and therefore breaks the operations that permute it with the other three. The subgroup of  $Fm\bar{3}m$  that preserves the selected  $\mathbf{k} = (1/2, 1/2, 1/2)$  is the index-four rhombohedral subgroup  $G = R\bar{3}m$  (#166). Keeping  $Fm\bar{3}m$  would instead require the four-dimensional star representation, which acts on the whole order-parameter space spanned by the four arms, including multi- $\mathbf{k}$  states, rather than on any one ordered state.

Within this reduced parent group the collinear order is described by a real one-dimensional representation at the finite

wave vector  $\mathbf{k}$ . Its translation character follows directly from the modulation of the antiferromagnetic state,

$$\Gamma([\mathbf{1}|\mathbf{t}]) = e^{i\mathbf{k} \cdot \mathbf{t}}. \quad (73)$$

Since the  $L$  point is a time-reversal invariant wave vector,  $2\mathbf{k} \equiv 0$ , these characters are real and take only the values  $\pm 1$ . In particular, a translation connecting neighboring ferromagnetic (111) planes changes the sign of the order parameter,

$$\Gamma([\mathbf{1}|\mathbf{t}_\perp]) = -1, \quad (74)$$

which encodes the alternating stacking of the type-II antiferromagnet. In the notation of the  $R\bar{3}m$  group this is the  $T$  point and the associated representation is the one-dimensional representation  $T_1^+$  in the Miller–Love convention [94], with no star since the star of  $T$  in  $R\bar{3}m$  has a single arm.

Several physical properties then follow without requiring the detailed point-group content of  $\Gamma$ . Since the representation is at finite wave vector, the order is compensated, with no linear piezomagnetism or time-odd spin conductivity (Section IV A). The time-even spin conductivity vanishes identically because the order is collinear. Thus all of these conclusions follow directly from the dimensionality and wave-vector content of the representation, without needing to assign a spin space group label. The spin texture also vanishes identically for MnO. The defining one-dimensional representation lives at the finite  $T$  point and therefore cannot overlap any  $\mathbf{k} = 0$  momentum multipole, while the collinear spin-only symmetry excludes the odd-momentum sector altogether.

### H. $\text{Gd}_3(\text{Ru}_{0.95}\text{Rh}_{0.05})_4\text{Al}_{12}$

The material  $\text{Gd}_3(\text{Ru}_{0.95}\text{Rh}_{0.05})_4\text{Al}_{12}$  is a Rh-substituted variant of the breathing-kagome magnet  $\text{Gd}_3\text{Ru}_4\text{Al}_{12}$  [138]. It crystallizes in space group  $P6_3/mmc$  (#194) [139] and hosts a commensurate coplanar spin helix below  $T_{N2} \approx 9$  K with propagation vector [140]

$$\mathbf{k} = \left(\frac{1}{6}, \frac{1}{6}, 0\right). \quad (75)$$

A single- $\mathbf{k}$  helix selects one of the three symmetry-related pairs  $\{\mathbf{k}, -\mathbf{k}\}$  in the hexagonal Brillouin zone. The full  $P6_3/mmc$  space group therefore cannot be the parent space group of its spin space group: acting with the full point group generates all three pairs and hence a six-dimensional star representation, as the full cubic group does for MnO (Section V G). Restricting instead to the subgroup that preserves the selected pair gives the index-three orthorhombic subgroup

$$G = Cmcm \quad (76)$$

(#63), in whose reciprocal-lattice coordinates the propagation vector is  $\mathbf{k} = (\frac{1}{3}, 0, 0)$ .

The origin of the two-dimensional representation is particularly transparent for a helix. Let  $\hat{N}_1$  and  $\hat{N}_2$  span the spin rotation plane. The magnetic structure may be written as

$$\mathbf{S}(\mathbf{r}) = S \left[ \hat{N}_1 \cos(\mathbf{k} \cdot \mathbf{r}) + \hat{N}_2 \sin(\mathbf{k} \cdot \mathbf{r}) \right]. \quad (77)$$

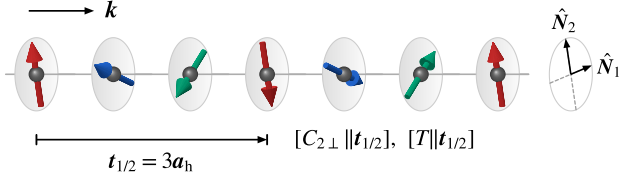


FIG. 11. The helix of  $\text{Gd}_3(\text{Ru}_{0.95}\text{Rh}_{0.05})_4\text{Al}_{12}$ , Eq. (77), on a row of Gd sites related by  $\mathbf{a}_h$ , drawn against position along  $\mathbf{k}$ . The moments turn in the plane normal to  $\mathbf{k}$  (disks). The phase advances by  $\pi/3$  per  $\mathbf{a}_h$ , so the translation  $\mathbf{t}_{1/2} = 3\mathbf{a}_h$  reverses every moment. This gives the  $p$ -wave symmetries  $[C_{2\perp} || \mathbf{t}_{1/2}]$  and  $[T || \mathbf{t}_{1/2}]$ .

We have idealized the experimentally refined structure by this circular single-harmonic helix, neglecting its small distortions and the accompanying magnetization,  $\approx 2\%$  of the saturated moment [140]. The pair of functions  $\{\cos(\mathbf{k} \cdot \mathbf{r}), \sin(\mathbf{k} \cdot \mathbf{r})\}$  therefore furnishes a real two-dimensional representation of  $Cmcm$ . Under a translation  $\mathbf{t}$  this basis transforms as

$$\Gamma([1|\mathbf{t}]) = \begin{pmatrix} \cos(\mathbf{k} \cdot \mathbf{t}) & -\sin(\mathbf{k} \cdot \mathbf{t}) \\ \sin(\mathbf{k} \cdot \mathbf{t}) & \cos(\mathbf{k} \cdot \mathbf{t}) \end{pmatrix}. \quad (78)$$

Thus the spatial phase advance of the helix is represented directly as a rotation in spin space. Since the phase of the helix is  $\mathbf{k} \cdot \mathbf{r}$  itself, an operation  $[R|\mathbf{t}]$  with  $R\mathbf{k} = \mathbf{k}$  acts in the same way as its translation alone, so the little group of  $\mathbf{k}$  carries the irrep in which every point operation acts trivially. In the Miller–Love notation [94] this is  $\Sigma_1$ , with  $\Sigma = (u, 0, 0)$  the line of  $Cmcm$  containing  $\mathbf{k}$ , and  $\Gamma$  is the two-dimensional star representation  $^*\Sigma_1$  induced from it.

For the commensurate state the magnetic unit cell is six chemical unit cells, so the translation  $\mathbf{t}_{1/2} = 3\mathbf{a}_h$  by three hexagonal lattice vectors has  $\mathbf{k} \cdot \mathbf{t}_{1/2} = \pi$ , and  $\Gamma([1|\mathbf{t}_{1/2}]) = -1$ , which in three-dimensional spin space is a two-fold rotation about the normal to the ordering plane (Fig. 11). Combined with the spin-only symmetry of coplanar order, this gives the half-period symmetries  $[C_{2\perp} || \mathbf{t}_{1/2}]$  and  $[T || \mathbf{t}_{1/2}]$  used to characterize  $p$ -wave magnets [21, 140]. The results below do not rely on them: they follow from  $\Lambda^2\Gamma$  for any coplanar  $\{\mathbf{k}, -\mathbf{k}\}$  order, commensurate or not (Section VIC 1).

Several consequences also follow directly from the representation. Since  $\Gamma$  is constructed entirely from the finite wave vectors  $\mathbf{k}$  and  $-\mathbf{k}$ , the order is compensated, with no linear piezomagnetism or time-odd spin conductivity (Section IV A). The time-even spin conductivity instead depends on the one-dimensional representation  $\det\Gamma$ , which can be determined without any further work. In the basis  $\{\cos(\mathbf{k} \cdot \mathbf{r}), \sin(\mathbf{k} \cdot \mathbf{r})\}$ , any operation  $g = [R|\mathbf{t}]$  with  $R\mathbf{k} \equiv -\mathbf{k}$  acts as

$$\Gamma(g) = \begin{pmatrix} \cos(\mathbf{k} \cdot \mathbf{t}) & \sin(\mathbf{k} \cdot \mathbf{t}) \\ \sin(\mathbf{k} \cdot \mathbf{t}) & -\cos(\mathbf{k} \cdot \mathbf{t}) \end{pmatrix} \quad (79)$$

and therefore has  $\det\Gamma(g) = -1$ , independently of the phase  $\mathbf{k} \cdot \mathbf{t}$  carried by its translation part. Since  $Cmcm$  is centrosymmetric and inversion sends  $\mathbf{k} \rightarrow -\mathbf{k}$ , we have

$$\det\Gamma(I) = -1, \quad (80)$$

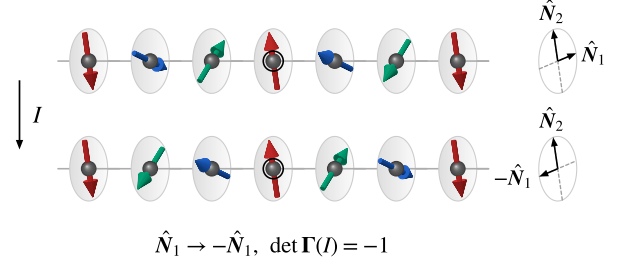


FIG. 12. The helix of Fig. 11 (top) and its image under inversion about the circled site (bottom), each moment carried with its site. The image is the original helix with  $\hat{N}_1 \rightarrow -\hat{N}_1$  (frames at right), a reflection in spin space, so  $\det\Gamma(I) = -1$ .

so  $\det\Gamma$  is odd under inversion (Fig. 12). Thus an operation that preserves  $\mathbf{k}$  acts as a rotation and has determinant +1, while one that reverses  $\mathbf{k}$  acts as a reflection and has determinant -1. The determinant is therefore the one-dimensional representation recording whether an operation preserves or reverses  $\mathbf{k}$ , which is precisely how the polar-vector component along  $\mathbf{k}$  transforms. Since  $\mathbf{k} \parallel \mathbf{a}$ , we have  $\det\Gamma = B_{3u}$  of  $D_{2h}$ , transforming as  $q_x$  in the convention with  $x \parallel \mathbf{a}$  and  $z \parallel \mathbf{c}$ . As  $\Gamma_0^+$ ,  $\Gamma_1^+$  and  $\Gamma_2^+$  are all even, there is no overlap and the time-even spin conductivity vanishes as well. The anomalous Hall conductivity vanishes since the order is coplanar, so the uniform magnetization, the piezomagnetic response, the spin conductivities and the anomalous Hall conductivity considered above are all forbidden in the non-relativistic limit.

The momentum-space spin texture, controlled by  $\Lambda^2\Gamma = \det\Gamma$ , is purely perpendicular to the ordering plane and begins at  $p$ -wave order. With  $\det\Gamma = B_{3u}$  transforming as  $q_x$ , this gives  $s(\mathbf{q})$  transforming as  $q_x(\hat{N}_1 \times \hat{N}_2)$  to leading order [Fig. 2(d)], while the even in-plane texture vanishes. Being the rank-one case, this also permits a time-even Edelstein response, by Section IVE. The contrast with the time-even spin conductivity is instructive: both are controlled by  $\det\Gamma$ , but the former requires overlap with the polar-vector representation  $\Gamma_1^-$  and the latter with the axial-vector representation  $\Gamma_1^+$ , so the inversion-odd  $\det\Gamma$  allows one and forbids the other.

## VI. DISCUSSION

Having illustrated the approach on specific materials, we now place it in a broader context. The same representation-first viewpoint applies to conventional magnetic space groups, each of which is encoded in a real one-dimensional representation of its parent space group (Section VIA). For collinear order, the defining representation is the Landau order-parameter irrep of the magnetic transition when that irrep is one-dimensional (Section VIB). We then use the exterior powers to characterize odd-parity magnets in general, including their wave-vector content and inversion parity (Section VIC), and compare the resulting texture conditions with existing spin space group classifications (Section VID). We close with the scope and limitations of the approach (Section VIE).

### A. Magnetic Space Groups

A similar representation-first viewpoint can be advocated for the usual magnetic space groups (MSGs), excluding the type II (grey) groups, which do not describe spontaneous breaking of time-reversal symmetry. Recall that each MSG defines a real one-dimensional irrep of the parent space group  $G$ ,

$$\Phi : G \rightarrow \mathbb{Z}_2, \quad (81)$$

with the sign indicating the presence or absence of time-reversal symmetry. These are subgroups of  $\mathbb{Z}_2 \times G$  instead of the  $O(3) \times G$  we have for the spin space groups. The corresponding operation for  $g \equiv [\mathbf{R}|\mathbf{t}]$  takes the form (in the spin space group notation)

$$[\Phi(g) \det(\mathbf{R}) \mathbf{R} \| g]. \quad (82)$$

This is the three-dimensional axial vector representation  $\Gamma_1^+$  times the one-dimensional irrep  $\Phi$ , which thus can be interpreted as a spin space group with  $\Gamma = \Phi \otimes \Gamma_1^+$ . As there is no spin-only group this should be interpreted within the non-coplanar framework. The usual classification of non-trivial magnetic space groups into types I, III, IV is then recovered by the following conditions on  $\Phi$ . Type I groups have  $\Phi = \Gamma_0^+$ , the trivial irrep. Type III groups have  $\Phi \neq \Gamma_0^+$  but  $\mathbf{k} = 0$ , while type IV groups have  $\mathbf{k} \neq 0$  [3].

Our results can be translated immediately to this language, and are collected in Table VII. A magnetic space group is compensated if and only if  $\Phi$  does not overlap with  $\Gamma_1^+$ . It is piezomagnetic if and only if  $\Phi$  overlaps with the pseudovector, quadrupole or octupole representations since  $\Gamma_1^+ \otimes \Gamma_2^+ = \Gamma_1^+ \oplus \Gamma_2^+ \oplus \Gamma_3^+$ , and the scalar part  $\Gamma_0^+$  of the strain contributes only a further  $\Gamma_1^+$ . For either quantity to be finite requires that  $\mathbf{k} = 0$ , i.e. that the magnetic space group is type I or type III. This condition is less prescriptive than it appears, since the sum of the three irreps  $\Gamma_1^+ \oplus \Gamma_2^+ \oplus \Gamma_3^+$  contains every (even) one-dimensional irrep of the point group except for the cubic groups  $O$ ,  $T_d$  and  $O_h$  where the trivial irrep is missing. Thus only type IV MSGs, type I MSGs with point group  $O$ ,  $T_d$  or  $O_h$ , and centrosymmetric type III MSGs where  $\Phi$  is inversion odd can fail to be piezomagnetic.

Next we can consider transport. Since  $\Phi^2$  and  $\det \Gamma_1^+$  are trivial,  $\det \Gamma = \det(\Phi \otimes \Gamma_1^+) = \Phi^3 \det \Gamma_1^+ = \Phi$ , so the time-even part of the spin conductivity is always allowed:  $\Gamma \otimes \det \Gamma = \Gamma_1^+$ . This is expected, as the time-even spin conductivity is allowed by symmetry even in free space so long as relativistic effects are included [56, 57, 59]. The time-odd part has a non-trivial, but fairly relaxed condition that  $\Phi$  overlaps with  $\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+ \oplus \Gamma_3^+$ . This requires  $\Phi$  to lie at  $\mathbf{k} = 0$ , so that the group is of type I or III, and for every crystallographic point group it holds whenever  $\Phi$  is even under inversion. Only type IV groups, and type III groups with inversion-odd  $\Phi$ , can therefore lack a time-odd spin conductivity.

Like in the spin space group case, the symmetric part of the thermal or charge conductivity is always allowed. The antisymmetric Hall part is allowed if  $\det \Gamma = \Phi$  overlaps with  $\Gamma_1^+$ , which is the same as the condition for lack of compensation. Note that  $\Gamma_1^+$  is purely three-dimensional for the cubic

point groups  $T$ ,  $T_h$ ,  $O$ ,  $T_d$  and  $O_h$ , and so contains no one-dimensional  $\Phi$ ; no magnetic space group with a cubic point group can therefore be uncompensated or exhibit an anomalous Hall response. This is a key difference from the spin-space group case, where the antisymmetric part of the charge conductivity can be allowed even for compensated magnets.

Substituting  $\Gamma = \Phi \otimes \Gamma_1^+$  and  $\det \Gamma = \Phi$  into the momentum-space spin-texture condition gives

$$\Gamma \otimes (\det \Gamma)^\ell = \Gamma_1^+ \otimes \Phi^{\ell+1}. \quad (83)$$

Since  $\Phi^2 = \Gamma_0^+$ , the conditions for even- and odd-rank textures separate. For even  $\ell \geq 2$ , a contribution is allowed when  $\Phi \otimes \Gamma_1^+$  overlaps with  $\Gamma_\ell^+$ , or equivalently when  $\Phi$  overlaps with  $\Gamma_{\ell-1}^+ \oplus \Gamma_\ell^+ \oplus \Gamma_{\ell+1}^+$ . For odd  $\ell$ , by contrast, the dependence on  $\Phi$  drops out entirely and the condition reduces to an overlap between  $\Gamma_1^+$  and  $\Gamma_\ell^-$ . Since  $\Gamma_1^+$  is even and  $\Gamma_\ell^-$  odd under inversion, this overlap is possible only in a point group without inversion. Thus even-in-momentum spin textures probe the magnetic representation  $\Phi$ , while odd-in-momentum textures are controlled entirely by the crystallographic symmetry.

The lowest ranks reproduce familiar response conditions. At  $\ell = 0$  the condition is precisely that for a net magnetization, while at  $\ell = 2$  it becomes

$$\Phi \subseteq \Gamma_1^+ \oplus \Gamma_2^+ \oplus \Gamma_3^+, \quad (84)$$

which is exactly the condition for piezomagnetism. Thus an MSG admits a  $d$ -wave spin texture if and only if it admits piezomagnetism, and higher even ranks extend this same magnetic-multipole hierarchy [29].

For type IV magnetic space groups,  $\Phi$  lives at finite wave vector, so all even-rank contributions are forbidden and only odd-in-momentum spin textures can occur. In a centrosymmetric parent group the spin texture therefore vanishes identically.

The Edelstein conditions reduce in the same way. The time-odd component requires  $\Phi$  to be contained in  $\Gamma_0^- \oplus \Gamma_1^- \oplus \Gamma_2^-$ , and thus to live at  $\mathbf{k} = 0$ , so only types I and III can exhibit it; in a centrosymmetric group  $\Phi$  must in addition be odd under inversion. The time-even component is the  $\ell = 1$  case of the odd-rank texture condition, and so loses its dependence on  $\Phi$  entirely, reducing to  $\Gamma_1^+ \cap \Gamma_1^- \neq 0$ . This condition is independent of the MSG and requires a non-centrosymmetric point group, but not every such group qualifies:  $\Gamma_1^-$  and  $\Gamma_1^+$  share no irrep in  $C_{3h}$ ,  $D_{3h}$  and  $T_d$ , as Table III shows. The remaining eighteen non-centrosymmetric point groups are the gyrotropic ones, and for those the time-even Edelstein response is permitted for every type.

### B. Collinear Magnets and Landau Theory

For collinear order the representation-first description is closely tied to an object that is already standard: the Landau order-parameter irrep of the magnetic transition, with which it coincides when that irrep is one-dimensional. This equivalence was implicit in McClarty and Rau [98] and demonstrated explicitly in Schiff *et al.* [99]. Schiff *et al.* [141] extend the

| Response                    |           | Symbol                   | Spin space group ( $\Gamma$ )                                            | Magnetic space group ( $\Phi$ )                                                | MSG type |     |    |
|-----------------------------|-----------|--------------------------|--------------------------------------------------------------------------|--------------------------------------------------------------------------------|----------|-----|----|
|                             |           |                          |                                                                          |                                                                                | I        | III | IV |
| Net spin magnetization      |           | $\mathbf{M}$             | $\Gamma \cap \Gamma_0^+$                                                 | $\Phi \cap \Gamma_1^+$                                                         | *        | *   | ×  |
| Piezomagnetism              |           | $\mathbf{C}_{\mu\nu}$    | $\Gamma \cap (\Gamma_0^+ \oplus \Gamma_2^+)$                             | $\Phi \cap (\Gamma_1^+ \oplus \Gamma_2^+ \oplus \Gamma_3^+)$                   | *        | *   | ×  |
| Spin conductivity           | Time-odd  | $\Sigma^-$               | $\Gamma \cap (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+)$           | $\Phi \cap (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+ \oplus \Gamma_3^+)$ | ✓        | *   | ×  |
|                             | Time-even | $\Sigma^+$               | $\Lambda^2 \Gamma \cap (\Gamma_0^+ \oplus \Gamma_1^+ \oplus \Gamma_2^+)$ | Always                                                                         | ✓        | ✓   | ✓  |
| Anomalous Hall conductivity |           | $\sigma^-$               | $\Lambda^3 \Gamma \cap \Gamma_1^+$                                       | $\Phi \cap \Gamma_1^+$                                                         | *        | *   | ×  |
| Edelstein effect            | Time-odd  | $\alpha^-$               | $\Gamma \cap \Gamma_1^-$                                                 | $\Phi \cap (\Gamma_0^- \oplus \Gamma_1^- \oplus \Gamma_2^-)$                   | *        | *   | ×  |
|                             | Time-even | $\alpha^+$               | $\Lambda^2 \Gamma \cap \Gamma_1^-$                                       | $\Gamma_1^+ \cap \Gamma_1^-$                                                   | *        | *   | *  |
| Spin texture                | Even rank | $\mathbf{s}(\mathbf{q})$ | $\Gamma \cap \Gamma_\ell^+$                                              | $(\Phi \otimes \Gamma_1^+) \cap \Gamma_\ell^+$                                 | *        | *   | ×  |
|                             | Odd rank  | $\mathbf{s}(\mathbf{q})$ | $\Lambda^2 \Gamma \cap \Gamma_\ell^-$                                    | $\Gamma_1^+ \cap \Gamma_\ell^-$                                                | *        | *   | *  |

TABLE VII. Symmetry conditions for each observable, stated in the spin space group language of Section IV and translated into the magnetic space group language of this section. A magnetic space group corresponds to  $\Gamma = \Phi \otimes \Gamma_1^+$ . Piezomagnetism is listed in its general form, which reduces to the  $\Gamma \cap \Gamma_2^+$  of Table I for compensated order. Types are as defined in the text,  $\Phi = \Gamma_0^+$  for type I,  $\Phi \neq \Gamma_0^+$  at  $\mathbf{k} = 0$  for type III, and  $\mathbf{k} \neq 0$  for type IV. The last three columns record whether the response is allowed for every magnetic space group of that type (✓), forbidden for that type whatever the point group (×), or allowed subject to the condition in the preceding column (\*).

Landau description to multi-dimensional irreps and to coplanar and non-coplanar order. We summarize the collinear argument here.

As set out in Eq. (4), a collinear spin space group is fixed entirely by a single real one-dimensional representation  $\Gamma$  of the parent space group  $G$ . The same representation appears if one instead writes down a Landau theory for the transition. In the non-relativistic limit the free energy is invariant under  $O(3) \times G_0$ , and the order parameter has one component for each dimension of the Landau irrep of  $G_0$ , each a vector in spin space. For collinear order these all lie along a common unit vector  $\hat{N}$ . Spin rotations act only on  $\hat{N}$ , so the entire spatial transformation law is carried by the amplitudes of the components, which are scalars under spin rotations. If the Landau irrep is one-dimensional there is a single amplitude  $\phi$ , every operation of  $G_0$  maps  $\phi$  to  $\pm\phi$ , and so  $G = G_0$  and the Landau irrep is  $\Gamma$  itself. If it is multi-dimensional, as for a wave-vector star with more than one arm or a two- or three-dimensional irrep at  $\mathbf{k} = 0$ , the ordered state selects a direction in the space of amplitudes, and  $\phi$  is the amplitude along that direction. The parent space group  $G < G_0$  is then the subgroup that maps  $\phi$  to  $\pm\phi$ , and  $\Gamma$  is the sign with which it does so. For MnO the Landau irrep is the four-dimensional star representation of the  $L$  point of  $Fm\bar{3}m$ , and  $\Gamma$  is the one-dimensional  $T_1^+$  of  $R\bar{3}m$  (Section VG).

The practical consequence is that the input required by the results of Sections VA, VB, VF and VG is precisely the output of a conventional representation analysis of the ordered state, as routinely performed when refining a magnetic structure [28]. Decomposing the permutation representation of the magnetic sites into irreps of  $G_0$ , the real one-dimensional irreps are the candidate regular collinear orders, with  $G = G_0$ , and each is a candidate  $\Gamma$ . This is the route taken in Sections VA and VB, where the permutation representations  $A_{1g} \oplus B_{2g}$  of  $\text{MnF}_2$  and  $A_{1g} \oplus B_{1g}$  of  $\text{MnTe}$  identify  $\Gamma$  immediately, with the trivial irrep in each case corresponding to the ferromagnet. The systematic development of these Landau theories, and their associated physical consequences for

collinear altermagnets, is given in Schiff *et al.* [99].

Seen this way, many of the boxed criteria of Section IV are statements about invariants of the Landau free energy. Compensation is the requirement that no term linear in the net magnetization is allowed, and the piezomagnetic condition that  $\Gamma$  overlaps  $\Gamma_2^+$  is the existence of a trilinear invariant coupling the order parameter, the strain, and the magnetization along  $\hat{N}$  – which is why the allowed coupling in  $\text{MnF}_2$  takes the form  $\mathbf{M} \propto \epsilon_{xy} \hat{N}$ . For collinear magnets, then, no spin space group need be constructed at any stage: the Landau irrep one already has gives the representation the construction requires, directly when it is one-dimensional and by restriction to  $G$  otherwise. For coplanar or non-coplanar order arising from a single irrep, however, that irrep is necessarily multi-dimensional, and the relation to the underlying spin space group is more delicate, as developed in Schiff *et al.* [141].

### C. Odd-Parity Magnets

We define odd-parity magnets (including the so-called  $p$ -wave magnets [21, 142]) as magnets whose momentum-space spin texture is purely odd under momentum inversion. [143] Textures of this kind have appeared in other contexts, notably in the theory of Fermi-liquid instabilities, where an odd-parity order parameter in the particle-hole spin channel spin-splits the Fermi surface without relativistic spin-orbit coupling [144–146]. Our representation-first approach allows a systematic characterization of this class of odd-parity magnets, in particular their wave-vector content and inversion parity, directly from the properties of the defining representation  $\Gamma$ .

Based on the results of Section IV D, collinear order cannot support any odd-rank spin texture, whatever the parent space group [21, 22], in the absence of other forms of time-reversal symmetry breaking [93, 147]. For coplanar and non-coplanar order alike the even-rank texture is controlled by  $\Lambda^1 \Gamma$  and the odd-rank texture by  $\Lambda^2 \Gamma$ , so a purely odd-parity texture

requires

$$\Lambda^1 \Gamma \cap \Gamma_\ell^+ = 0 \quad (85)$$

for all even  $\ell$ , while

$$\Lambda^2 \Gamma \cap \Gamma_\ell^- \neq 0 \quad (86)$$

for at least one odd  $\ell$ . The smallest odd  $\ell$  for which the second condition is satisfied defines the wave character of the magnet:  $\ell = 1$  gives a  $p$ -wave magnet,  $\ell = 3$  an  $f$ -wave magnet, and so on through the odd multipoles. These conditions place strong constraints on the representation  $\Gamma$  and hence on the spin space group. Which irreps can appear in a momentum multipole of given rank and parity is established in Appendix B.

The rank-one condition,  $\Lambda^2 \Gamma \cap \Gamma_1^- \neq 0$ , is the condition for a time-even linear Edelstein response found in Section IV E. Among odd-parity magnets, the  $p$ -wave ones are thus exactly those in which an electric field can generate a time-even spin polarization at linear order, a non-relativistic charge-to-spin conversion [92]. Those whose lowest odd rank is  $f$ -wave or higher admit no such response, as found for the  $h$ -wave texture of coplanar order in the Fe-based superconductors [148].

### 1. Coplanar Odd-Parity Magnets

Consider coplanar order, for which the defining representation  $\Gamma$  is two-dimensional and

$$\Lambda^2 \Gamma = \det \Gamma. \quad (87)$$

A purely odd-parity spin texture requires that  $\Gamma$  have no overlap with any even-rank momentum multipole, while  $\det \Gamma$  must overlap with at least one odd-rank multipole. Since  $\Lambda^2 \Gamma = \det \Gamma$  is also the representation carried by the vector spin chirality  $\mathbf{S}_1 \times \mathbf{S}_2$ , this second condition connects directly with the odd-parity order parameter of Yu *et al.* [149].

We first constrain the wave-vector content. To have an odd-parity texture  $\Lambda^2 \Gamma = \det \Gamma$  must have a  $\mathbf{k} = 0$  component and so the wave-vector content must be  $\{\mathbf{k}, -\mathbf{k}\}$  including the special case of two components at the same TRIM,  $\mathbf{k} \equiv -\mathbf{k} \equiv \mathbf{K}$ . The case  $\mathbf{k} = 0$  cannot produce a purely odd-parity texture. If the parent point group does not contain inversion, every irrep occurs in some even-rank momentum multipole, so  $\Gamma$  necessarily has an even-rank overlap. If inversion is present, absence of every even-rank contribution requires every component of  $\Gamma$  to be odd under inversion. But for a two-dimensional representation this makes  $\det \Gamma$  even under inversion, and hence unable to overlap any odd-rank multipole  $\Gamma_\ell^-$ . If  $\mathbf{k} \neq 0$  then the even-rank overlap is automatically forbidden, independent of the presence of inversion, and the appearance of the odd-parity texture is controlled by the point group. In short, a coplanar purely odd-parity magnet must have wave-vector content  $\{\mathbf{k}, -\mathbf{k}\}$  with  $\mathbf{k} \neq 0$ .

The remaining condition is a point-group one:

$$\det \Gamma \cap \Gamma_\ell^- \neq 0 \quad (88)$$

for at least one odd  $\ell$ . For the non-centrosymmetric case ( $P$  does not contain inversion) only the wave-vector content is

relevant, since every irrep appears in some odd-rank multipole and in some even-rank multipole. This is the case for the brownmillerite  $\text{Ca}_2\text{Cr}_2\text{O}_5$  [24, 150], whose  $\Gamma = 2X_1$  of  $Ima2$  has  $\Lambda^2 \Gamma = \det \Gamma = A_1$ , the polar-axis component of  $\Gamma_1^-$ , and so a  $p$ -wave texture. In the centrosymmetric case ( $P$  contains inversion) the point-group condition is non-trivial. For a pair  $\{\mathbf{k}, -\mathbf{k}\}$  where  $\mathbf{k} \neq -\mathbf{k}$  inversion only swaps  $\mathbf{k}$  and  $-\mathbf{k}$  (it is not in the little group of  $\mathbf{k}$ ) and so  $\det \Gamma(I) = -1$  and hence it overlaps some odd-rank multipole. For two components at the same TRIM  $\{\mathbf{K}, \mathbf{K}\}$  inversion can act non-trivially and allow or forbid the odd-rank overlap depending on the inversion parity of  $\det \Gamma$ . Both outcomes can occur. For example, in  $\text{GeNi}_2\text{O}_4$  [24, 151] the two components of  $\Gamma = T_1^+ \oplus T_2^-$  of  $R\bar{3}m$  have opposite inversion parity, so that  $\Lambda^2 \Gamma = \det \Gamma = A_{2u}$  and the texture is  $p$ -wave, whereas in  $\text{LiFeGe}_2\text{O}_6$  [24, 152]  $\Gamma = Y_1^+ \oplus Y_2^+$  of  $P2_1/c$  has  $\Lambda^2 \Gamma = \det \Gamma = B_g$  and no spin texture at all. [153]

To summarize, a coplanar order with  $\{\mathbf{k}, -\mathbf{k}\}$ ,  $\mathbf{k} \neq 0$ , has a purely odd-parity texture except at a repeated TRIM in a centrosymmetric parent, where  $\det \Gamma$  must also be odd under inversion. We collect this result, together with the non-coplanar results below, in Table VIII.

Further, for a non-TRIM pair we can obtain a stronger result, as the wave character is fixed by  $\mathbf{k}$  alone. An operation that preserves  $\mathbf{k}$  acts on the pair as a rotation and one that reverses it as a reflection (see Section V H), so  $\det \Gamma$  records only whether  $\mathbf{Rk} \equiv \mathbf{k}$  or  $\mathbf{Rk} \equiv -\mathbf{k}$ , irrespective of the irrep at  $\mathbf{k}$  the order transforms as. For  $\mathbf{k}$  in the interior of the Brillouin zone, no reciprocal lattice vector is needed in these conditions:  $\mathbf{Rk}$  and  $\pm \mathbf{k}$  are then both interior points, which cannot differ by a nonzero reciprocal lattice vector, so  $\mathbf{Rk} = \pm \mathbf{k}$  exactly. Hence  $\det \Gamma$  transforms identically to  $\mathbf{k} \cdot \mathbf{q}$ . The magnet is thus  $p$ -wave and admits a time-even Edelstein response. Its leading texture is  $(\mathbf{c} \cdot \mathbf{q}) \hat{N}_1 \times \hat{N}_2$ , where  $\mathbf{c}$  lies in the subspace of polar vectors transforming as  $\det \Gamma$ . This subspace always contains  $\mathbf{k}$ . When  $\det \Gamma$  occurs only once in  $\Gamma_1^-$ , as for  $B_{3u}$  of  $D_{2h}$  in  $\text{Gd}_3(\text{Ru}_{0.95}\text{Rh}_{0.05})_4\text{Al}_{12}$ ,  $\mathbf{c}$  is therefore parallel to  $\mathbf{k}$ . For a parent with point group  $C_i$ , by contrast,  $\det \Gamma = A_u$  occurs three times in  $\Gamma_1^-$ , and  $\mathbf{c}$  can point in any direction. Higher odd waves therefore require  $\mathbf{k}$  on the zone boundary: at the  $K$  point of a hexagonal crystal with point group  $D_{6h}$ , for instance,  $\det \Gamma$  is an inversion-odd  $B$  irrep and the leading texture is  $f$ -wave. This is the case for the  $120^\circ$  order of the triangular antiferromagnet  $\text{CsMnBr}_3$ , for which  $\det \Gamma = B_{1u}$  [24, 154].

**Coplanar Odd-Parity Magnets:** A coplanar order whose wave-vector content is a non-TRIM pair  $\{\mathbf{k}, -\mathbf{k}\}$  has a purely odd-parity spin texture, as does one with two components at the same nonzero TRIM, provided that in a centrosymmetric parent  $\Gamma(I)$  has eigenvalues of both signs. For a non-TRIM  $\mathbf{k}$  in the interior of the Brillouin zone, the texture is  $p$ -wave and a time-even Edelstein response is allowed. Higher odd waves, such as  $f$ -wave, require  $\mathbf{k}$  on the zone boundary or a repeated TRIM.

This construction includes the original  $p$ -wave construction of Hellenes *et al.* [21], but does not in general require the

| Order        | Content                                      | Without $I$                  | With $I$                                                              |
|--------------|----------------------------------------------|------------------------------|-----------------------------------------------------------------------|
| Coplanar     | $\{\mathbf{k}, -\mathbf{k}\}$                | always                       | always                                                                |
|              | $\{\mathbf{K}, \mathbf{K}\}$                 | always                       | both parities                                                         |
| Non-coplanar | $\{\mathbf{k}, -\mathbf{k}, \mathbf{K}\}$    | $\mathbf{K} \neq \mathbf{0}$ | odd at $\mathbf{K} = \mathbf{0}$                                      |
|              | $\{\mathbf{K}_1, \mathbf{K}_1, \mathbf{K}\}$ | $\mathbf{K} \neq \mathbf{0}$ | both parities at $\mathbf{K}_1$ ,<br>odd at $\mathbf{K} = \mathbf{0}$ |
|              | $\{\mathbf{K}, \mathbf{K}, \mathbf{K}\}$     | always                       | both parities                                                         |

TABLE VIII. Conditions on  $\Gamma$  for a purely odd-parity spin texture, by its wave-vector content, for parent point groups without and with inversion  $I$ . Here  $\mathbf{k}$  is not a TRIM,  $\mathbf{K}$  and  $\mathbf{K}_1$  are distinct TRIMs, and a repeated TRIM is nonzero. “Both parities” requires  $\Gamma(I)$  to have eigenvalues of both signs on the repeated components. “Odd at  $\mathbf{K} = \mathbf{0}$ ” requires any component at  $\mathbf{K} = \mathbf{0}$  to be odd under  $I$ . No other wave-vector content admits a purely odd-parity texture.

ordering wave vector to admit a half-period lattice translation. Such a translation exists when there is a lattice vector  $\mathbf{t}_{1/2}$  satisfying

$$e^{i\mathbf{k} \cdot \mathbf{t}_{1/2}} = -1. \quad (89)$$

Every nonzero TRIM has this property, as do commensurate non-TRIM wave vectors whose translation character has even order. Certain incommensurate wave vectors may also admit such a translation, but its existence is not guaranteed. The representation criterion above, by contrast, only requires the pair  $\{\mathbf{k}, -\mathbf{k}\}$  and therefore applies more generally [155], including to experimentally realized incommensurate odd-parity magnets such as  $\text{NiI}_2$  [156]. [157]

## 2. Non-Coplanar Odd-Parity Magnets

Consider non-coplanar order, for which the defining representation  $\Gamma$  is three-dimensional. Odd-parity spin textures arising from non-coplanar magnetic order have also been identified directly from spin-group symmetry [158]. A purely odd-parity spin texture again requires that  $\Gamma$  have no overlap with any even-rank momentum multipole, while  $\Lambda^2\Gamma$  must overlap with at least one odd-rank multipole.

We first constrain the wave-vector content. For  $\Lambda^2\Gamma$  to have a  $\mathbf{k} = \mathbf{0}$  component  $\Gamma$  must have wave-vector content of the form  $\{\mathbf{k}, -\mathbf{k}, \mathbf{K}\}$ , where  $\mathbf{K}$  is a TRIM (see Section III). The case  $\mathbf{k} = \mathbf{0}$  cannot produce a purely odd-parity texture. If inversion is absent, the two  $\mathbf{k} = \mathbf{0}$  components necessarily produce an even-rank overlap. If inversion is present, removing this overlap requires both to be odd under inversion. Their exterior product is then even. The remaining parts of  $\Lambda^2\Gamma$ , which involve the third component, either lie at finite wave vector, if  $\mathbf{K} \neq \mathbf{0}$ , or are also even, if  $\mathbf{K} = \mathbf{0}$ , since that component must then be odd as well. Either way no odd-rank texture results.

We thus assume that  $\mathbf{k} \neq \mathbf{0}$ . For a non-centrosymmetric parent, the remaining condition is simply  $\mathbf{K} \neq \mathbf{0}$ , since a component at  $\mathbf{K} = \mathbf{0}$  would overlap some even-rank multipole. All components of  $\Gamma$  then lie at finite wave vector, while  $\Lambda^2\Gamma$  has a  $\mathbf{k} = \mathbf{0}$  component, the exterior square of the cancelling

pair. Since every irrep appears in some odd-rank multipole, this is sufficient for a purely odd-parity texture.

For a centrosymmetric parent, a component of  $\Gamma$  at  $\mathbf{K} = \mathbf{0}$  is allowed provided it is odd under inversion. Such a component can also overlap  $\Gamma_1^-$ , so that, unlike an order built entirely from finite wave vectors, the magnet may then show a time-odd Edelstein response. The cancelling pair  $\{\mathbf{k}, -\mathbf{k}\}$  must also produce an inversion-odd component of  $\Lambda^2\Gamma$ . For a non-TRIM pair  $\mathbf{k} \neq -\mathbf{k}$  this is automatic, since inversion exchanges the two sectors. If instead the cancelling pair consists of two components at the same nonzero TRIM  $\mathbf{K}_1$ , those two components must have opposite inversion parity.

There is one additional special case when all three components occur at the same nonzero TRIM, so that the wave-vector content is  $\{\mathbf{K}, \mathbf{K}, \mathbf{K}\}$ . Then every pair contributes at  $\mathbf{k} = \mathbf{0}$  to  $\Lambda^2\Gamma$ . An odd-rank texture is allowed provided inversion does not act with the same parity on all three components: equivalently,  $\Gamma(I)$  must contain both  $+1$  and  $-1$  eigenvalues.[159]

The wave character then follows as in the coplanar case. Outside the case  $\{\mathbf{K}, \mathbf{K}, \mathbf{K}\}$ , the  $\mathbf{k} = \mathbf{0}$  part of  $\Lambda^2\Gamma$  is the exterior square of the cancelling pair alone, so the odd texture points along the spin direction of the  $\mathbf{K}$  component and its rank is set by that one-dimensional representation. For a non-TRIM  $\mathbf{k}$  inside the Brillouin zone it is therefore again  $p$ -wave, with a time-even Edelstein response, while higher odd waves require  $\mathbf{k}$  on the zone boundary or a cancelling pair at a TRIM.

These conditions are collected, with those for coplanar order, in Table VIII.

## D. Spin Texture Classifications

Momentum-space textures of the kind derived in Section IV D have been analyzed directly using spin space groups, including the symmetry of non-relativistic electronic bands in Liu *et al.* [20], the even-wave spin splitting of collinear altermagnets [6], and the odd-parity texture perpendicular to the ordering plane of coplanar  $p$ -wave magnets [21]. Song *et al.* [22] have since given a systematic spin space group classification of these textures. The parity relations of Eqs. (35) and (36a) to (36c) appear there, together with a classification of the collinear and coplanar wave hierarchies. For a texture that is constrained to a fixed spin axis, they define  $\ell$  as the lowest polynomial degree whose representation contains the one-dimensional representation associated with that texture.

The distinction from the present viewpoint is therefore not the representation-theoretic character of this criterion, but where the required representation is obtained. In the spin space group approach it is determined from an identified spin space group, while here it is already determined by the defining representation, as  $\Lambda^1\Gamma$  for collinear order and  $\Lambda^1\Gamma$  or  $\Lambda^2\Gamma$  for coplanar and non-coplanar order. Moreover, because no reduction to a fixed spin axis is required, the same overlap condition applies directly to vector-valued spin textures, with  $\Lambda^1\Gamma$  controlling the even and  $\Lambda^2\Gamma$  the odd ranks. Thus the momentum-space multipole content follows from the parent space group and its defining representation, with the alternation of even and odd ranks between  $\Lambda^1\Gamma$  and  $\Lambda^2\Gamma$  visible

directly.

### E. Scope and Limitations

Three important limitations bound the core results of this work.

The first is the non-relativistic limit itself: couplings that lock spin to space are invisible here by construction, which is why the anomalous Hall effects observed in MnTe [110] and Mn<sub>3</sub>Ge [118, 119] require relativistic interactions or extrinsic perturbations that break the protecting spin-only symmetry. Our conditions are therefore statements about what survives as relativistic effects are switched off at fixed magnetic structure, and are most useful as a leading-order account to which relativistic corrections are added. When relativistic effects are included in full, the symmetry is instead that of a magnetic space group, which the same representation-first approach also covers, though less cleanly, through  $\Gamma = \Phi \otimes \Gamma_1^+$  (Section VIA).

The second is the kind of question asked. The exterior-power conditions derived here are overlaps of a power of  $\Gamma$  with a representation of the point group, and so fix the symmetry-allowed form of a response: which components of a tensor may be nonzero, or which multipoles a momentum-space texture may contain. A tensor allowed in this sense may still vanish in particular models, through a conservation law that the full problem need not share. The uniform time-odd Edelstein response of a collinear magnet, for example, vanishes when the carrier spin is conserved, as for Bloch electrons in a fixed exchange field, but not once the carriers exchange spin with the ordered moments (Section IVE). Nor is every equilibrium property an overlap of this kind.

The anisotropy of the static spin susceptibility, at fixed orientation of the order in spin space, is governed instead by the invariant symmetric bilinear forms on  $\Gamma$ , that is, by the number of times  $\Gamma_0^+$  occurs in the symmetric square  $\text{sym}^2(\Gamma)$ , together with the spin-only symmetry. For collinear and coplanar orders, the spin-only symmetry singles out the Néel axis or the ordering plane, allowing anisotropic susceptibilities. By contrast, a non-coplanar order whose  $\Gamma$  is irreducible, as for FeF<sub>3</sub> and CoTa<sub>3</sub>S<sub>6</sub>, has an isotropic susceptibility enforced by Schur's lemma. For such properties the symmetric square plays the role that the exterior powers play for the responses treated here.

These conditions also do not fix the symmetry labels of individual states. Where the electronic or magnon bands are degenerate, how those labels connect across the Brillouin zone, and which of their topological invariants symmetry can diagnose are constrained by the little co-groups of the spin space group at each crystal momentum, including its antiunitary elements and the translations they carry; answering them means constructing at least those groups [9, 10, 160–162], and a character overlap in  $G$  does not suffice.

The third is that every quantity treated is uniform, and every response linear. The responses are unit-cell averages, and although the spin texture depends on  $\mathbf{q}$ , its multipole moments [Eq. (33)] average it over the Brillouin zone, so that each, like the responses, transforms under a representation of  $G$  at  $\mathbf{k} = \mathbf{0}$ .

Sublattice-resolved or locally staggered responses, such as a local Edelstein effect, can transform under other representations of  $G$  and lie outside the present analysis. Nonlinear responses, by contrast, are mostly still uniform and should follow in the same way from overlaps of powers of  $\Gamma$  with the appropriate multipole representations. We leave them to future work.

## VII. CONCLUSIONS

We have argued that the symmetry consequences of a magnetic order in the non-relativistic limit are best read from the real representation  $\Gamma$  of its parent space group, rather than from the spin space group it generates. This spares assembling and identifying the group, but its main value is structural. The responses considered here reduce to overlaps between the three exterior powers of  $\Gamma$  and low-rank multipole representations, with the power fixed by the presence of a spin index and by time-reversal parity, and the spin-only symmetries of collinear and coplanar order entering through the vanishing of the higher powers. The same powers impose a hierarchy of wave-vector selection rules (one ordering wave vector vanishing, two cancelling, or all three summing to zero) and control the even- and odd-rank momentum-space spin textures, of which the time-even Edelstein effect is the rank-one case. They also delimit odd-parity magnets: a purely odd-parity texture requires a cancelling pair of nonzero wave vectors, and in coplanar order an interior non-TRIM pair always gives a  $p$ -wave texture.

The examples of Sections VA to VH show how little is often needed. For MnO the dimension and wave-vector content of the representation settle every response without any point-group label at all: a representation at finite  $\mathbf{k}$  cannot overlap anything at  $\mathbf{k} = \mathbf{0}$ , and collinearity disposes of the rest. For Gd<sub>3</sub>(Ru<sub>0.95</sub>Rh<sub>0.05</sub>)<sub>4</sub>Al<sub>12</sub> the time-even responses follow from identifying  $\det \Gamma$  with the polar-vector component along the propagation direction, without needing the full irrep label of  $\Gamma$ . Where the point-group content does matter it separates materials that are otherwise alike: MnF<sub>2</sub> and Cr<sub>2</sub>O<sub>3</sub> are both compensated collinear antiferromagnets, but only the first is piezomagnetic, and the distinction is visible immediately as  $B_{2g} \subset \Gamma_2^+$  against  $A_{2u} \not\subset \Gamma_2^+$ . The examples also show where care is needed: when the order selects part of a wave-vector star, the parent group need not be the crystallographic space group, and using the latter would force a star representation of the wrong dimension.

The same construction is not peculiar to spin space groups. A magnetic space group is encoded by a real one-dimensional representation  $\Phi$  of its parent, from which the conventional type I, III and IV classification and all of the response conditions follow in the same way, and for collinear order in the non-relativistic limit  $\Gamma$  is the Landau order-parameter irrep of the transition when that irrep is one-dimensional, and otherwise follows from it by restriction to the parent space group of the ordered state. The representation the construction requires is therefore one that a standard analysis of an ordered structure already produces. This is the practical content of the distinction we have drawn between classification and inference. A classification remains necessary to enumerate symmetry types,

compare settings, and assign canonical labels. Going from a magnetic structure to its symmetry-enforced properties can instead be framed directly in terms of the representation, where the connections between different responses, and their origin in the ordered state, are made explicit.

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### Appendix A: Exterior Powers of a Direct Sum

The exterior powers of a reducible representation are fixed by those of its summands. Let  $\Gamma = \Gamma_a \oplus \Gamma_b$  act on  $V = V_a \oplus V_b$ , where  $\Gamma_a$  and  $\Gamma_b$  are themselves representations of  $G$ . A wedge of  $n$  vectors of  $V$  expands into terms carrying  $p$  factors from  $V_a$  and  $q = n - p$  from  $V_b$ , and  $\Gamma(g)$  preserves this grading, so

$$\Lambda^n \Gamma = \bigoplus_{p+q=n} \Lambda^p \Gamma_a \otimes \Lambda^q \Gamma_b \quad (\text{A1})$$

as representations of  $G$  [34]. Iterating this gives

$$\Lambda^n \left( \bigoplus_i \Gamma_i \right) = \bigoplus_{\sum_i p_i = n} \bigotimes_i \Lambda^{p_i} \Gamma_i. \quad (\text{A2})$$

Each  $\Gamma_i$  must be a representation of  $G$  in its own right; splitting  $V$  into subspaces that the group mixes does not decompose  $\Lambda^n \Gamma$ .

Since  $\Lambda^p \Gamma_i$  vanishes for  $p$  greater than the dimension of  $\Gamma_i$ , few terms survive in low dimensions. When every summand is one-dimensional only  $p_i = 0, 1$  contribute and

$$\Lambda^n \left( \bigoplus_{i=1}^d \Gamma_i \right) = \bigoplus_{i_1 < \dots < i_n} \Gamma_{i_1} \otimes \dots \otimes \Gamma_{i_n}, \quad (\text{A3})$$

the sum of the products of all  $n$  distinct summands.

The reducible two- and three-dimensional cases follow, with  $^d \Gamma$  denoting a  $d$ -dimensional irreducible representation as in Section IV C 2. For  $\Gamma = {}^1 \Gamma \oplus {}^1 \Gamma'$ ,

$$\Lambda^2 \Gamma = {}^1 \Gamma \otimes {}^1 \Gamma', \quad (\text{A4a})$$

$$\Lambda^3 \Gamma = 0. \quad (\text{A4b})$$

For  $\Gamma = {}^1 \Gamma \oplus {}^2 \Gamma$ ,

$$\Lambda^2 \Gamma = ({}^1 \Gamma \otimes {}^2 \Gamma) \oplus \det({}^2 \Gamma), \quad (\text{A5a})$$

$$\Lambda^3 \Gamma = {}^1 \Gamma \otimes \det({}^2 \Gamma). \quad (\text{A5b})$$

For  $\Gamma = {}^1 \Gamma \oplus {}^1 \Gamma' \oplus {}^1 \Gamma''$ ,

$$\Lambda^2 \Gamma = ({}^1 \Gamma \otimes {}^1 \Gamma') \oplus ({}^1 \Gamma \otimes {}^1 \Gamma'') \oplus ({}^1 \Gamma' \otimes {}^1 \Gamma''), \quad (\text{A6a})$$

$$\Lambda^3 \Gamma = {}^1 \Gamma \otimes {}^1 \Gamma' \otimes {}^1 \Gamma''. \quad (\text{A6b})$$

In each case the top power reproduces  $\Lambda^n \Gamma = \det \Gamma$ , as it must for any  $n$ -dimensional representation, reducible or not.

### Appendix B: Parity Content of Momentum Multipoles

Here we justify a fact used in Section VI C. The decomposition of spherical harmonics into irreducible representations of point groups is standard [163]; we reproduce the argument here in the form we require. Let  $P$  be a crystallographic point group and choose a generic direction  $\hat{q}$  whose stabilizer in  $P$  is trivial. Its orbit  $P\hat{q}$  therefore has one point for each element of  $P$ , so arbitrary functions on this orbit transform as the regular representation and contain every irrep of  $P$ .

Suppose first that inversion is not contained in  $P$ . For a generic choice of  $\hat{q}$ , the orbit  $P\hat{q}$  and its antipodal copy  $-P\hat{q}$  are disjoint. An even function on their union is specified by arbitrary values on  $P\hat{q}$ , with the values on the antipodal orbit fixed to be equal. An odd function is specified in exactly the same way, with the antipodal values fixed to have the opposite sign. Thus both the even and odd functions contain the regular representation of  $P$ , and hence every irrep of  $P$ . Since every function on a finite set of directions is the restriction of a finite sum of spherical harmonics, with even and odd functions requiring only even and odd ranks respectively, every irrep occurs in some even-rank multipole and in some odd-rank multipole.

If inversion  $I$  is contained in  $P$ , the antipodal point already belongs to the same orbit. Inversion commutes with every element of the point group, so each irrep has a definite inversion parity. Splitting the functions on  $P\hat{q}$  into those that are even or odd under  $\hat{q} \rightarrow -\hat{q}$  therefore splits the regular representation into its inversion-even and inversion-odd parts. Consequently, every inversion-even irrep occurs in some even-rank multipole, while every inversion-odd irrep occurs in some odd-rank multipole.

### Appendix C: Extracting the Defining Representation

For a commensurate magnetic structure, choose a supercell on which the operations of  $G_0$  act by permuting its  $N$  magnetic sites. Express the moments in an orthonormal basis of the spin subspace they span, and collect them as the columns of a matrix  $\mathbf{S} \in \mathbb{R}^{n \times N}$  of rank  $n$ , where  $n = 1, 2, 3$  for collinear, coplanar and non-coplanar order.

For an operation  $g$ , let  $\mathbf{P}_g$  be the permutation matrix such that column  $i$  of  $\mathbf{S}\mathbf{P}_g$  is the moment at site  $g(i)$ . A compensating orthogonal spin transformation exists if and only if  $g$  preserves every pairwise scalar product of the moments, that is, the Gram matrix  $\mathbf{S}^\top \mathbf{S}$ ,

$$\mathbf{P}_g^\top \mathbf{S}^\top \mathbf{S} \mathbf{P}_g = \mathbf{S}^\top \mathbf{S}. \quad (\text{C1})$$

Since the columns of  $\mathbf{S}$  span the occupied spin subspace, this transformation is then unique,

$$\mathbf{\Gamma}(g) = \mathbf{S}\mathbf{P}_g \mathbf{S}^\top (\mathbf{S}\mathbf{S}^\top)^{-1}, \quad (\text{C2})$$

and it is orthogonal and satisfies  $\mathbf{S}\mathbf{P}_g = \mathbf{\Gamma}(g)\mathbf{S}$ . The operations satisfying Eq. (C1) form the parent space group  $G$ , and their matrices  $\mathbf{\Gamma}(g)$  give its defining representation. For  $n < 3$  the spin-only symmetries are added as described in Section II.

The response analysis then proceeds by restricting  $\mathbf{\Gamma}$  and its exterior powers to the translations, identifying their components at  $\mathbf{k} = 0$ , and testing the overlaps listed in Table I.

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